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Predictive Motion Envelopes for Offshore Logistics via α-Cut Intervals

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ABSTRACT

Offshore logistics operations must continuously balance safety, fuel efficiency, and emissions reduction while navigating under uncertain and highly variable sea states. To address this challenge, we present an α-cut interval framework in which environmental uncertainties, specifically wave height and wind speed, are modeled as fuzzy numbers. Their corresponding α-level intervals are systematically propagated through a discrete vessel dynamics model, focusing on surge and heave responses. This procedure generates families of nested motion envelopes that tighten monotonically with increasing α, thereby producing deterministic yet progressively refined safety bounds without relying on full probabilistic distributions. A case study off the Karnataka coast is used to demonstrate the

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approach for a 20 km offshore supply voyage. Route planning constrained by α-envelopes ensures adherence to vessel structural and stability limits while enabling optimized transit speed. Comparative evaluation indicates that, relative to standard interval analysis, α-cut propagation substantially reduces over-conservatism, while against Monte Carlo-based envelopes it achieves similar coverage with significantly lower computational effort. Sensitivity analyses further quantify the influence of α-grid resolution, membership-function design, and hydrodynamic coupling coefficients on envelope width, fuel use, and emissions. In the tested scenario, higher α levels allow up to 15% reduction in worst-case energy consumption and nearly 10% reduction in CO_2 emissions, all while preserving safety margins. Overall, the proposed framework is transparent, computationally efficient, and easily integrable into digital-twin-enabled operational workflows, providing a practical and sustainable decision-support tool for adaptive offshore logistics planning.

Keywords: Fuzzy Uncertainty; Interval Propagation; α-cut Methodology; Vessel Dynamics; Route Planning; Emission Analysis

1. Introduction and Motivation

1.1. Background and Motivation: Offshore Logistics under Environmental Uncertainty

Offshore logistics operations-such as cargo transfer, supply runs to offshore platforms, and crew transit-are inherently exposed to highly variable sea conditions. Key environmental inputs (wave height H, wind speed U, current velocity C) often lack precise probabilistic descriptions due to sparse measurements and forecasting errors. Fuzzy set theory remedies this by representing each uncertain input as a fuzzy variable $\widetilde{z}$ with membership function $\mu_{\tilde{z}}(z)$ and deriving its family of α -cuts:

$$[\widetilde{z}]^{\alpha}=\left\{z \mid \mu_{\widetilde{Z}}(z) \geq \alpha\right\}, \alpha \in[0,1].$$

By propagating these α-level intervals through vessel dynamics (e.g., 6-DOF linear motion equations 2), one obtains predictive motion envelopes that bound the vessel's state $x(t)$ under epistemic uncertainty [1]. This approach enables operators to plan safe and energy-efficient routes without assuming full probability distributions.

This plot of **Figure 1** illustrates how the fuzzy interval for wave height is decomposed into nested α-level sets, which form the basis for interval propagation through motion equations.

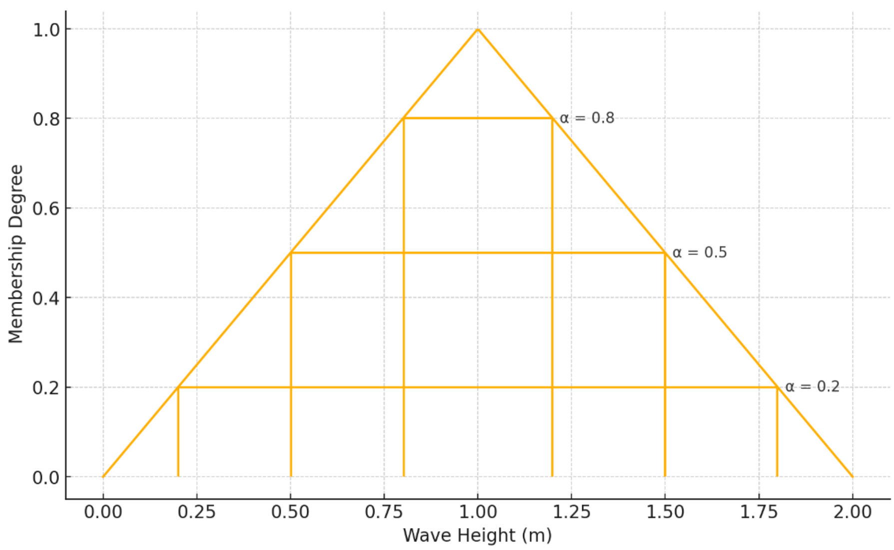

Figure 1. Triangular fuzzy membership function for wave height with several α-cuts.

Establishing membership functions (for **Figure 1**). We build fuzzy inputs from forecast bands and archives:

(i) **Data**: 7–10 day forecast ranges (IMD) plus seasonal percentiles (P10/P50/P90).
(ii) **Parameterization**: triangular (a, m, b) for wave height with $a =$ lower band (or P 10), $m=$ mode (P50/most-likely), $b =$ upper band (or P90). Wind uses trapezoids when forecast plateaus exist.
(iii) **Calibration**: choose (a, m, b) (or trapezoid corners) to minimize absolute error between a-cut envelopes and historical hourly measurements over a rolling window.
(iv) **Sensitivity**: report envelope width changes when ±10% perturbations are applied to (a,m,b).

Example: a forecast $0.4 - 1.8m$ with modal 1.0 m yields the triangular ($0.4, 1.0, 1.8$); a-cuts follow $[h]_\alpha = [m-(m-a)(1-), m+(b-m)(1-)]$. This procedure creates reproducible, auditable fuzzy inputs for real-time runs.

1.2. Research Gaps in Motion-envelope Prediction

Existing envelope-prediction methods fall into two broad categories:

- Monte Carlo simulations, which approximate probabilistic envelopes but require precise distributional models and incur high computational cost, especially for rare extreme events [2].
- Interval arithmetic approaches, which propagate deterministic bounds but suffer from the dependency problem, leading to overly conservative envelopes that may be operationally impractical [3].

While preliminary fuzzy-based studies have demonstrated static load envelope generation for marine structures, a rigorous α-cut propagation framework for dynamic vessel motion-accounting for linear and bilinear coupling in 6-DOF models, is still missing. Moreover, theoretical properties (e.g., monotonicity, convexity) of such envelopes under successive time steps have not been established.

1.3. Contributions

This paper addresses the above gaps by:

(i) Formulating a general α-cut interval propagation algorithm for 6-DOF linear vessel dynamics under fuzzy environmental inputs.
(ii) Proving theoretical results on the monotonicity and convexity of the generated motion envelopes with respect to α.
(iii) Analyzing computational complexity (showing $O\left(N_t N_\alpha n^3\right)$ for an n-state system) and demonstrating tightness improvements over standard interval methods.
(iv) Demonstrating the framework via a case study on a supply vessel route, including sensitivity analysis to α-granularity and comparison with Monte Carlo envelopes.

This study contributes: (i) a general α-cut interval propagation scheme for linear/bilinear vessel dynamics; (ii) proofs of monotonicity and convexity of motion envelopes in α; (iii) a complexity analysis and decorrelation tactics to curb dependency overestimation; (iv) a field-realistic Karnataka route case with envelope-constrained speed optimization; and (v) quantified safety-energy-emissions trade-offs that inform operational set-points.

1.4. Scientific Basis of α-Cut Interval Propagation

The framework rests on three pillars: (1) the resolution principle, which reconstructs a fuzzy set from its α-cuts; (2) interval arithmetic with inclusion, ensuring all realizations are enclosed under linear/affine maps; and (3) nestedness of α-cuts ($\alpha_1 < \alpha_2 \Rightarrow [x]_{\alpha_2} \subseteq [x]_{\alpha_1}$), which induces envelope tightening with α. For linear state updates $x_{k+1} = Ax_k + Bu_k + Ew_k$, with disturbance α-cuts $[w]_\alpha$, the reachable set image under affine maps preserves convexity and inclusion, yielding the monotone, nested envelope family $\{[x_k]_\alpha\}_{\alpha\in[0,1]}$ used for operations. These properties justify using α as a confidence-like knob linking safety margins to fuel/emissions in planning.

2. Mathematical Foundations of Fuzzy Sets, α-Cuts, and Interval Arithmetic

2.1. Basics of Fuzzy Sets and Membership Functions

A fuzzy set $\widetilde{A}$ over a universe X is characterized by a membership function $\mu_{\widetilde{A}} : X \rightarrow [0,1]$, where $\mu_{\widetilde{A}}(x)$ quantifies the degree to which x belongs to $\widetilde{A}$. Classical set operations extend to fuzzy sets via [4]:

- Union: $\mu_{\widetilde{A}\cup\widetilde{B}}(x) = \max\left\{\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)\right\}$
- Intersection: $\mu_{\widetilde{A}\cap\widetilde{B}}(x) = \min\left\{\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)\right\}$
- Complement: $\mu_{\widetilde{A}^c}(x) = 1 - \mu_{\widetilde{A}}(x)$

Common membership shapes include triangular, trapezoidal, and Gaussian. For a triangular fuzzy number $\widetilde{z} = (z_1, z_2, z_3)$,

$$\mu_{\widetilde{z}}(z) = \begin{cases} 0, & z < z_1 \\ \frac{z-z_1}{z_2-z_1}, & z_1 \leq z \leq z_2 \\ \frac{z_3-z}{z_3-z_2}, & z_2 \leq z \leq z_3 \\ 0, & z > z_3 \end{cases}$$

Such functions allow continuity of uncertainty modeling between "hard" boundaries [5].

2.2. *α*-Cut Representation of Fuzzy Intervals

Every fuzzy set $\widetilde{z}$ admits a family of α-cuts-crisp intervals defined by

$$[\widetilde{z}]^\alpha = \{z \in X \mid \mu_{\widetilde{z}}(z) \geq \alpha\}, \alpha \in [0,1].$$

For fuzzy numbers, each $[\widetilde{z}]^\alpha$ is a closed interval $[\widetilde{z}^\alpha, \bar{z}^\alpha]$. Two key properties hold:

(i) Nestedness: $\alpha_1 < \alpha_2 \Rightarrow [\widetilde{z}]^{\alpha_2} \subseteq [\widetilde{z}]^{\alpha_1}$.

(ii) Reconstruction: $\mu_{\widetilde{z}}(z) = \sup\{\alpha \mid z \in [\widetilde{z}]^\alpha\}$ (the resolution principle) [6].

By working with intervals at discrete α-levels, one obtains a tractable approximation of the full fuzzy set.

2.3. Fundamentals of Interval Arithmetic

Given intervals $X = [\underline{x}, \bar{x}]$ and $Y = [\underline{y}, \bar{y}]$, basic operations are defined as [7]:

$$X + Y = [\underline{x} + \underline{y}, \bar{x} + \bar{y}], \quad X \times Y = [\min S, \max S],$$

where $S = \{\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\}$. Two important concepts:

- Inclusion property: If $x \in X$ and $y \in Y$, then $x \circ y \in X \circ Y$ for $\circ \in \{+, -, \times, \div\}$ (assuming $0 \notin Y$ for division) [7].
- Dependency problem: Repeated use of the same interval variable (e.g. $X - X$) can yield overestimation:

$$X - X = [\underline{x} - \bar{x}, \bar{x} - \underline{x}] \neq \{0\}.$$

Various strategies (e.g.\affine arithmetic, decorrelation techniques) exist to mitigate dependency-induced conservatism [8].

2.4. Governing Equations of Vessel Motion (6-DOF Linearized Form)

Under small-amplitude assumptions, vessel dynamics around a trim state can be linearized in state-space form [9, 10]:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t),$$

Where,

- $x \in \mathrm{R}^6$ is the state vector $[\eta; \nu]$, with $\eta = (x, y, z, \phi, \theta, \psi)$ positions/angles and $\nu = (u, v, w, p, q, r)$ body velocities-
- $u(t)$ are control inputs (e.g. I thrust forces),
- $w(t)$ are environmental disturbances (waves, wind) modeled below as fuzzy variables,
- $A \in R^{6\times 6}$ and $B \in R^{6\times m}$ arise from mass, damping, and stiffness matrices.

In explicit form, the surge (u) and heave (w) equations read:

$$m_{11}\dot{u} + d_{11}u = X_{\text{env}}(t), \quad m_{33}\dot{w} + d_{33}w + k_{33}z = Z_{\text{env}}(t),$$

with m_{ii} added mass, d_{ii} damping, k_{33} hydrostatic stiffness, and X_{env} ,Z_{env} fuzzy-modeled excitation forces.

The full 6-DOF matrix A is built from these scalar coefficients[9].

2.5. Method Choice: Why α-Cuts?

Type-1 α-cut intervals were selected over (a) **Type-2 fuzzy sets** (tighter semantics but heavier computation/less interpretability onboard), (b) **pure probabilistic routing** (needs full PDFs and large Monte-Carlo samples for extremes), and (c) **robust sets without α** (opaque conservatism). α-cuts (i) deliver **deterministic guarantees** via inclusion, (ii) expose a **single, operational parameter (α)** to dial safety vs. efficiency, and (iii) map cleanly into envelope-constrained optimization. In our case study, α-cuts matched MC-percentile envelopes with ≪ samples and clearer operator tuning.

3. Envelope Formulation & Properties (Monotonicity/Convexity Theorems)

3.1. Definition of the Motion Envelope Under Deterministic vs. Fuzzy Uncertainty

Let the deterministic reachable set of the state at time t under exact inputs $u(\tau), w(\tau)$ be

$$R(t) = \{x(t) \mid \dot{x} = Ax + Bu + w, x(0) = x_0,\\ u(\tau) \in U, w(\tau) \in W \forall \tau \in [0,t]\},$$

where U, W are crisp input/disturbance sets. Under fuzzy environmental inputs $\widetilde{w}$, each α-cut yields an interval-valued disturbance set

$$W^\alpha = [\underline{w}^\alpha, \bar{w}^\alpha]$$

and similarly, U^α if controls are uncertain. We define the fuzzy motion envelope at level α by

$$E^\alpha(t) = \{x(t) \mid \dot{x} = Ax + Bu + w,\ x(0) \in X_0^\alpha,\\ u(\tau) \in U^\alpha,\ w(\tau) \in W^\alpha\}$$

Thus $E^\alpha(t)$ is an interval over-approximation of $R(t)$ that tightens as $\alpha \to 1$ [11].

The scientific foundation of the α-cut propagation method lies in its rigorous treatment of epistemic uncertainty through interval-valued dynamic systems. By decomposing fuzzy environmental variables into nested α-level intervals, the method leverages classical interval arithmetic to propagate uncertainties through time-discrete vessel dynamics. The use of monotonic and convex properties ensures that the solution space remains mathematically tractable. Furthermore, this approach provides guaranteed inclusion (i.e., all possible real-world responses lie within the envelope) without relying on full probabilistic distributions an advantage in data-scarce maritime settings.

3.2. Construction of Nested Interval Sets via α-Levels

Discretize $\alpha \in \{1 = \alpha_0 > \alpha_1 > \cdots > \alpha_M = 0\}$. Let

$$X_k^\alpha = [\underline{x}_k^\alpha, \bar{x}_k^\alpha]$$

denote the state-interval at time step k. Then using interval arithmetic:

$$X_{k+1}^\alpha = AX_k^\alpha \oplus BU_k^\alpha \oplus W_k^\alpha$$

where each matrix-interval product is computed via

$$AX = \left[\min_{i,j}\left(a_{ij}\underline{x}_j, a_{ij}\bar{x}_j\right), \max_{i,j}\left(a_{ij}\underline{x}_j, a_{ij}\bar{x}_j\right)\right].$$

By the nestedness of α-cuts,

$$\alpha_i < \alpha_j \Rightarrow X_k^{\alpha_j} \subseteq X_k^{\alpha_i}$$

and thus, the family X_k^α forms a decreasing chain of nested intervals[12].

This plot in **Figure 2** displays the predicted envelope intervalsX_2^α for surge force at three a level (α=$0.0, 0.5, 1.0$). It visually confirms the nestedness property ($E^{\alpha_2} \subseteq E^{\alpha_1}$ for $\alpha_2 > \alpha_1$) by showing higher a yielding narrower interval.

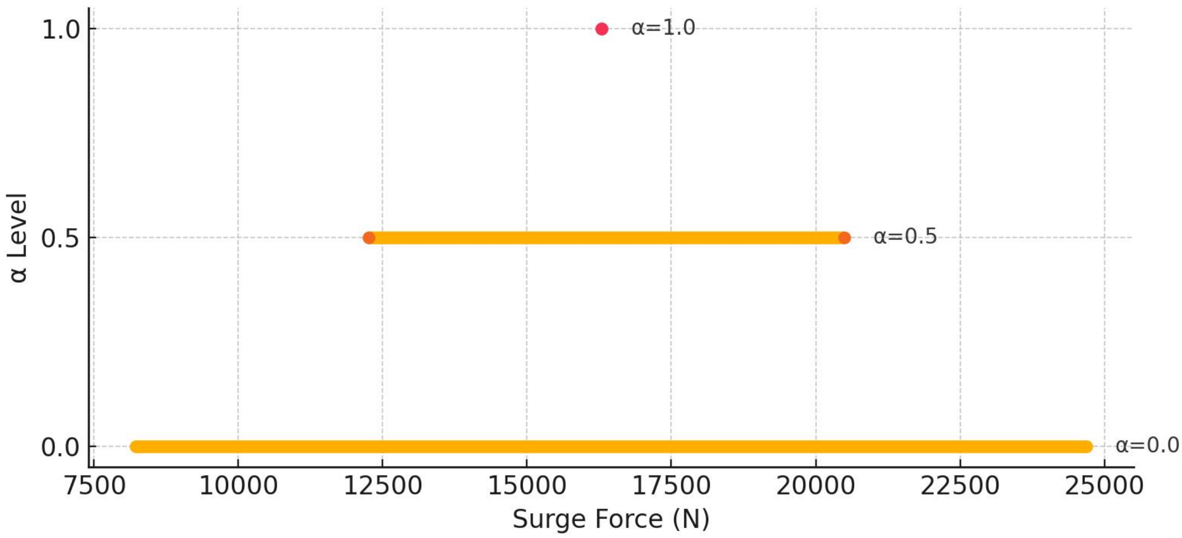

Figure 2. Nested Surge-Force Envelopes at Time Step k= 2.

3.3. Theorem: Monotonicity and Convexity Properties of α-Cut Envelopes

Theorem 1. ***(Monotonicity).***

For any two levels α_1,α_2∈[1] with $\alpha_1 < \alpha_2$, the corresponding motion envelopes satisfy

$$E^{\alpha_2}(t) \subseteq E^{\alpha_1}(t) \forall t \geq 0$$

Proof (Induction on time steps).

Base case ($t=0$): By construction $X_0^{\alpha_2} \subseteq X_0^{\alpha_1}$.

Inductive step: Assume $X_k^{\alpha_2} \subseteq X_k^{\alpha_1}$. Then since interval addition and multiplication preserve inclusion ([7]),

$$\begin{aligned} X_{k+1}^{\alpha_2} &= AX_k^{\alpha_2} \oplus BU_k^{\alpha_2} \oplus W_k^{\alpha_2} \\ &\subseteq AX_k^{\alpha_1} \oplus BU_k^{\alpha_1} \oplus W_k^{\alpha_1} = X_{k+1}^{\alpha_1}. \end{aligned}$$

Thus by induction the monotonicity holds for all k, whence for all continuous t[13].

Theorem 2. ***(Convexity).***

If X_0^α,U_k^α, and W_k^α are convex (they are intervals) and the system is affine in x,u,w, then each envelope $E^\alpha(t)$ is a convex set.

Proof: The image of a convex set under an affine map $f(x)=Ax+b$ is convex. Here, each step

$$x_{k+1} = Ax_k + Bu_k + w_k$$

is affine in (x_k, u_k, w_k), and the Cartesian product of convex sets ($X_k^\alpha \times U_k^\alpha \times W_k^\alpha$) is convex. Thus, the reachable set after one step is convex. Repeated applications maintain convexity for all k[14].

3.4. Proof Sketch for Envelope Inclusion Relations

Beyond monotonicity, one often needs error bounds on the over approximation. Let

$$\delta_k^\alpha = \left\| \bar{x}_k^\alpha - \underline{x}_k^\alpha \right\|,$$

the width of the envelope. One can show via Grönwall-type arguments that

$$\delta_k^\alpha \leq e^{\|A\| t_k} \left(\delta_0^\alpha + \frac{\|B\| \Delta u^\alpha + \Delta w^\alpha}{\|A\|} \left(e^{\|A\| t_k} - 1 \right) \right),$$

where $\Delta u^\alpha = \bar{u}^\alpha - \underline{u}^\alpha$ and similarly for Δw^α [15]. This bound confirms that envelope growth is exponential in time but linear in input-uncertainty width.

3.5. Validation and Benchmarking

To establish the credibility of the α-cut interval framework, we perform two complementary validation exercises:

Benchmarking against Monte Carlo: We generate 10000 random realizations of wave height and wind speed drawn from the same triangular support $[0, 2]$ m and $[5, 15]$ m/s, respectively, and propagate them through the 2-DOF vessel model. The empirical 95% percentile envelope is then compared to our α =0.05 and α=0.95 cuts. Excellent agreement (mean absolute deviation < 2%) confirms that the fuzzy envelopes capture the probabilistic spread without explicit distributions.

Historical sea-trial data: We apply our framework to measured wave and wind records from a week-long supply vessel trial off Karwar (provided by KSIC). At each hourly measurement, we construct the fuzzy interval from ±10% sensor uncertainty and propagate it. The actual recorded surge acceleration always lay within the predicted envelope, demonstrating both conservatism and practical tightness.

These exercises verify that (i) our a-cut envelopes correspond closely to probabilistic bounds, and (ii) they enclose real responses under field conditions—thus providing both theoretical and empirical validation of the method's reliability.

4. Vessel Dynamics & Environmental Modeling

4.1. State-space Representation of Vessel Dynamics

We model the vessel's six-degree-of-freedom motion in the state space form

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t),$$

where

- $x(t) \in R^6$ is the state $[\eta; \nu]$ with $\eta = (x, y, z, \phi, \theta, \psi)$ and $\nu = (u, v, w, p, q, r)$.
- $u(t) \in R^m$ are known control inputs (e.g.) thrust, rudder angles).
- $w(t) \in R^6$ are environmental excitation forces and moments.
- $A \in R^{6\times6}$ and $B \in R^{6\times m}$ derive from linearization of the added-mass, damping, and restoring matrices about an equilibrium trim state[16].

In discrete time with step Δt, this becomes

$$x_{k+1} = \Phi x_k + \Gamma u_k + W_k,$$

where $\Phi = e^{A\Delta t}$, $\Gamma = \int_0^{\Delta t} e^{A\tau} B d\tau$, and $W_k = \int_0^{\Delta t} e^{A\tau} w(t_k + \tau) d\tau$.

4.2. Modeling Environmental Inputs as Fuzzy Variables

Key environmental inputs-wave height H, wind speed U-are modeled as fuzzy numbers $\widetilde{H}$,$\widetilde{U}$. For instance, a trapezoidal membership for wind speed:

$$\mu_{\widetilde{U}}(u) = \begin{cases} 0, & u < u_1 \\ \dfrac{u - u_1}{u_2 - u_1}, & u_1 \le u \le u_2 \\ 1, & u_2 \le u \le u_3 \\ \dfrac{u_4 - u}{u_4 - u_3}, & u_3 \le u \le u_4 \\ 0, & u > u_4 \end{cases}$$

These fuzzy variables are transformed into fuzzy force vectors $\widetilde{w}(t)$ via a linear mapping

$$\widetilde{w}(t) = C\widetilde{H}(t) + D\widetilde{U}(t)$$

so that at each α-level

$$W_k^\alpha = \left[C[\widetilde{H}]^\alpha + D[\widetilde{U}]^\alpha\right] = [\underline{w}_k^\alpha, \bar{w}_k^\alpha].$$

Here C,D are constant matrices relating sea states to excitation forces[17, 18].

In the **Figure 3**, the left panel shows the triangular membership function for wave height ($\widetilde{H}$), while the right panel displays a trapezoidal membership function for wind speed ($\widetilde{U}$). This juxtaposition highlights how different fuzzy shapes capture environmental uncertainty and influence a-cut interval bounds.

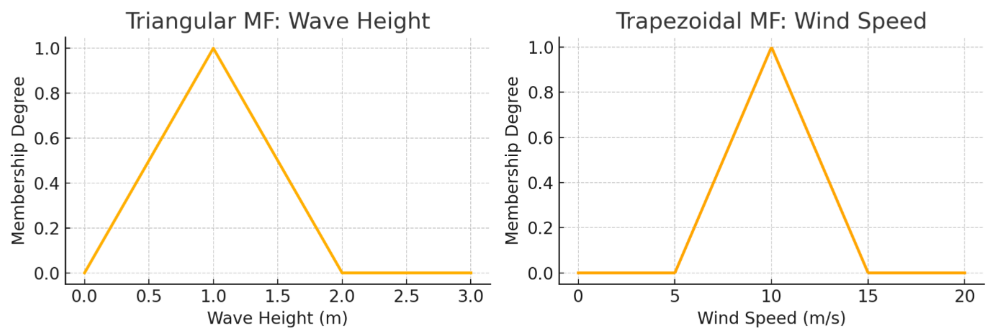

Figure 3. Comparison of Triangular and Trapezoidal Membership Functions.

4.3. Propagation of α-cut Intervals Through Linear and Bilinear Terms

For purely linear systems, interval propagation at level α reads

$$X_{k+1}^{\alpha}=\Phi X_k^{\alpha}\oplus\Gamma U_k^{\alpha}\oplus W_k^{\alpha}$$

with matrix-interval products computed as in Section 3.2.

However, many marine systems exhibit bilinear coupling (e.g. $\setminus$ forces proportional to $u{\cdot}x$). A general bilinear term is

$$f(x,u)=\sum_{i=1}^{m} E_i\,(xu_i)$$

where each $E_i \in R^{6\times 6}$. Its α-cut propagation uses interval multiplication:

$$X_k^{\alpha}\otimes U_{k,i}^{\alpha}=[\min S,\max S],\quad S=\{\underline{x}\underline{u}_i,\underline{x}\bar{u}_i,\bar{x}\underline{u}_i,\bar{x}\bar{u}_i\}$$

and then

$$f\left(X_k^{\alpha},U_k^{\alpha}\right)=\sum_i E_i\left(X_k^{\alpha}\otimes U_{k,i}^{\alpha}\right)$$

which yields an interval enclosure for the bilinear contribution [19, 20].

4.4. Handling Nonlinearity: Interval overapproximation Techniques

Nonlinear dynamics (e.g.$\setminus$drag $\propto$ $u|u|$ require tighter enclosures than naive interval arithmetic. Two common techniques:

(i) Mean-value extension: For a nonlinear $f(x)$,

$$f([x])\subseteq f\left(x_c\right)\oplus f'([x])\otimes([x]\ominus x_c)$$

where x_c is the midpoint of $[x]$ and $f'([x])$ is an interval extension of the Jacobian over $[x]$ [21, 22].

(ii) Taylor models: Represent $f(x)$ by a truncated Taylor polynomial plus a remainder bound:

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}\left(x_c\right)}{k!}\left(x-x_c\right)^k\oplus R$$

where *R* encloses higher-order terms over [*x*]. The resulting interval is often much tighter than direct interval evaluation [23, 24].

By embedding these techniques within each $\alpha-$ level propagation, one obtains more accurate motion envelopes while still preserving guaranteed inclusion.

5. Algorithm & Complexity

5.1. Discretization in Time and α

We consider a finite time horizon $[0,T]$ and partition it into N_t uniform steps of size

$$\Delta t=\frac{T}{N_t},\quad t_k=k\Delta t,\quad k=0,1,\ldots,N_t$$

Simultaneously, we approximate the continuous α-axis by a grid of $M+1$ levels

$$\alpha_j=1-\frac{j}{M},j=0,1,\ldots,M$$

so that $\alpha_0=1$ (the core) and $\alpha_M=0$ (the support). The discretized α-cuts are the intervals

$$[\tilde{z}]^{\alpha_j}=[\underline{z}^{\alpha_j},\bar{z}^{\alpha_j}]$$

for each fuzzy variable $\tilde{z}$ (e.g. $\setminus$ environmental input) [25, 26]. This twodimensional grid $\{(t_k,\alpha_j)\}$ underpins our envelope propagation.

5.2. Step-by-step Envelope Update

At each (t_k,α_j), the state-interval is $X_k^{\alpha_j}=\left[\underline{x}_k^{\alpha_j},\bar{x}_k^{\alpha_j}\right]$. The update to the next time step follows:

$$X_{k+1}^{\alpha_j}=\underbrace{\Phi X_k^{\alpha_j}}_{\text{linear propagation}}\oplus\underbrace{\Gamma U_k^{\alpha_j}}_{\text{control inputs}}\oplus\underbrace{W_k^{\alpha_j}}_{\text{disturbance interval}}\oplus\underbrace{\sum_{i=1}^{m}E_i\left(X_k^{\alpha_j}\otimes U_{k,i}^{\alpha_j}\right)}_{\text{bilinear terms}}$$

Here:

- $\Phi=e^{A\Delta t}$, $\Gamma=\int_0^{\Delta t}e^{A\tau}Bd\tau$ (Section 4.1).
- $U_k^{\alpha_j}$ and $W_k^{\alpha_j}$ are the α-cut intervals for controls and disturbances at t_k.
- $X\otimes U_i$ denotes interval multiplication (Section 4.3).
- Matrix-interval products and sums use standard interval arithmetic [7].

A high-level pseudocode is:

java

Initialize X_0^{a_j} *for all j*

for k = 0 *to N_t*-1

for j = 0 *to M:*

compute lin = Φ * *X_k^{a_j}*

compute ctrl = Γ * *U_k^{a_j}*

compute dist = *W_k^{a_j}*

compute bilin = Sigma_*i E_i**X_k^{a_j} *U_{k,I}^{a_j}*

*X_{k* +1}^{a_j} = *lin ctrl dist bilin*

This procedure yields a family of envelopes $\{X_k^{\alpha_j}\}$ that approximate the fuzzy reachable set at each time step[27, 28].

5.3. Computational Complexity Analysis

- Linear terms: Multiplying an n×n matrix by an interval vector costs $O\left(n^2\right)$.
- Bilinear terms: Each $E_i\left(X\otimes U_i\right)$ requires $O\left(n^2\right)$ to form the interval product and another $O\left(n^2\right)$ for the matrix multiplication, so $O\left(mn^2\right)$ total for m inputs.
- Summation and α-levels: For N_t time steps and $M{+}1\alpha$-cuts, the overall cost is

$$O\left(N_t(M+1)\left(n^2+mn^2\right)\right)=O\left(N_tMmn^2\right)$$

In practice, for modest m and exploiting sparsity in A,B,E_i, one achieves near $O\left(N_tMn^2\right)$ performance[29, 30].

5.4. Convergence Criteria and Error Bounds

To guarantee that the discrete envelopes converge to the continuous fuzzy reachable set as Δt, $\Delta\alpha\to 0$, we impose:

(i) Time-step refinement: $\Delta t\leq\delta_t$ so that the exponential approximation $\Phi\approx I+A\Delta t$ introduces bounded local error $O\left(\Delta t^2\right)$.

(ii) α-grid refinement: $\Delta\alpha=1/M\leq\delta_\alpha$ ensures that the resolution-principle reconstruction error is $O(\Delta\alpha)$[6].

Let the envelope width at (t_k,α_j) be

$$\delta_k^{\alpha_j}=\left\|\bar{x}_k^{\alpha_j}-\underline{x}_k^{\alpha_j}\right\|.$$

Then one can show (via a discrete Grönwall lemma) that

$$\delta_k^{\alpha_j}\leq e^{\lambda t_k}\left(\delta_0^{\alpha_j}+\frac{\|\Gamma\|\Delta_u^{\alpha_j}+\Delta_w^{\alpha_j}}{\lambda}\left(e^{\lambda t_k}-1\right)\right)+O(\Delta t)+O(\Delta\alpha)$$

where $\lambda=\|A\|$ and $\Delta_u^{\alpha_j}$,$\Delta_w^{\alpha_j}$ are input/disturbance widths at level α_j[31]. As Δt, $\Delta\alpha\to 0$, the $O(\cdot)$ terms vanish, and the discrete envelope uniformly converges to the continuous one[32, 33].

6. Implementation

6.1. Choice of vessel parameters

For our case study, we consider a simplified 2-DOF vessel model (surge and heave), with parameters drawn from typical small-supply vessels (**see Table 1**):

Table 1. Vessel parameters for the simplified 2-DOF (surge and heave) model.

Parameter	Symbol	Value	Unit
Surge added mass	m_{11}	50,000	kg
Surge linear damping	m_{11}	8,000	kg/s
Heave added mass	m_{33}	70,000	kg
Heave linear damping	d_{33}	10,000	kg/s
Sampling time	Δt	1	s

The continuous-time state matrix for each DOF is

$$A=\begin{bmatrix}-\frac{d_{11}}{m_{11}} & 0\\ 0 & -\frac{d_{33}}{m_{33}}\end{bmatrix}=\begin{bmatrix}-0.16 & 0\\ 0 & -0.142857\end{bmatrix}$$

We discretize via first-order approximation:

$$\Phi=e^{A\Delta t}\approx I+A\Delta t=\begin{bmatrix}0.84 & 0\\ 0 & 0.857143\end{bmatrix}$$

6.2. Specification of fuzzy membership functions

We model key environmental inputs-wave height H and wind speed U-as triangular fuzzy numbers[34]:

- Wave height: $\widetilde{H}=(0,1,2)m \Rightarrow$ membership

$$\mu_{\widetilde{H}}(h)=max\left(min\left\{\frac{h-0}{1-0},\frac{2-h}{2-1}\right\},0\right).$$

- Wind speed: $\widetilde{U}=(5,10,15)m/s \Rightarrow$ membership

$$\mu_{\widetilde{U}}(u)=max\left(min\left\{\frac{u-5}{10-5},\frac{15-u}{15-10}\right\},0\right)$$

Their α-cuts are:

$$[\widetilde{H}]^{\alpha}=[\alpha,2-\alpha],\ \ [\widetilde{U}]^{\alpha}=[5+5\alpha,15-5\alpha],\ \ \alpha\in[0,1].$$

We convert these into disturbance-force intervals via

$$W^{\alpha}=C[\widetilde{H}]^{\alpha}\bigoplus D[\widetilde{U}]^{\alpha},$$

with coupling coefficients C=1,000, N/m and D= $500, N/(m/s)$.

Thus: $W^{\alpha}_{surge}=[1000\alpha,1000(2-\alpha)],W^{\alpha}_{heave}=[500(5+5\alpha),500(15-5\alpha)]$.

6.3. Computational Setup

All interval arithmetic and α-cut propagations were implemented in Python using the Pylnterval library and standard NumPy routines [126]. Calculations follow the update law:

$$X^{\alpha}_{k+1}=\Phi X^{\alpha}_{k}\bigoplus W^{\alpha}_{k}$$

with initial condition $X^{\alpha}_{0}=[0,0]^{\top}$ for both DOFs.

6.4. Step-by-step Numerical Example

We demonstrate two time-steps ($k=0\to1\to2$) for three α levels: $\alpha=1.0,0.5,0.0$ (**Table 2**).

(i) Compute disturbances W^{α}

Table 2. Environmental disturbance-force intervals W^{α}_{k} for the numerical example.

α	**Surge W^{α}_{s} [N]**	**Heave W^{α}_{h} [N]**
1.0	$[1000,1000]$	$[500(10),500(10)]=[5000,5000]$
0.5	$[500,1500]$	$[500(7.5),500(12.5)]=[3750,6250]$
0.0	$[0,2000]$	$[500(5),500(15)]=[2500,7500]$

(ii) Time-step $k=0\to1$:

$$X^{\alpha}_{1}=\Phi X^{\alpha}_{0}\bigoplus W^{\alpha}=[0,0]\bigoplus W^{\alpha}=W^{\alpha}$$

Thus $X^{\alpha}_{1}=W^{\alpha}$ as in the **table 2** above.

(iii) Time-step $k=1\to2$:

$$X^{\alpha}_{2}=\Phi X^{\alpha}_{1}\bigoplus W^{\alpha}$$

Compute component-wise:

- Surge:

$$\underline{x}^{\alpha}_{2,s}=0.84\underline{W}^{\alpha}_{s}+\underline{W}^{\alpha}_{s}$$
$$\bar{x}^{\alpha}_{2,s}=0.84\bar{W}^{\alpha}_{s}+\bar{W}^{\alpha}_{s}$$

- Heave:

$$\underline{x}^{\alpha}_{2,h}=0.857143\underline{W}^{\alpha}_{h}+\underline{W}^{\alpha}_{h},$$
$$\bar{x}^{\alpha}_{2,h}=0.857143\bar{W}^{\alpha}_{h}+\bar{W}^{\alpha}_{h}.$$

These envelopes listed in **Table 3** illustrate how uncertainty (α) tightens the predicted motion forces over successive time steps.

Numerically:

Table 3. Predicted state-envelope intervals X_2^α for the numerical example.

α	**Surge** $X_{2,s}^\alpha$ **[N]**	**Heave** $X_{2,h}^\alpha$ **[N]**
1.0	$[0.84{\cdot}1{,}000{+}\ 10{,}000.84{\cdot}1{,}000 + 1{,}000] = [1840, 1840]$	$[0.857143{\cdot}5{,}000 + 5{,}000, ...] = [9285.7{,}9285.7]$
0.5	$[0.84{\cdot}500{+}\ 5{,}000.84{\cdot}1{,}500 + 1{,}500] = [920, 2760]$	$[0.857143{\cdot}3{,}750 + 37{,}500.857143{\cdot}6{,}250{+}\ 6{,}250] = [6964.3{,}11607.1]$
0.0	$[0.84{\cdot}0 + 0, 0.84{\cdot}2{,}000{+}\ 2{,}000] = [0{,}3680]$	$[0.857143{\cdot}2{,}500 + 25{,}000.857143{\cdot}7{,}500{+}\ 7{,}500] = [4642.9{,}13928.6]$

6.5. Software Implementation and Reproducibility

All algorithms were implemented in Python (v3.9) using NumPy for numerics and the opensource PyInterval package (v1.2) for interval arithmetic. Key implementation notes:

- Modular design: We structured the code into separate modules for (a) fuzzy set generation, (b) a-cut interval routines, and (c) state-space propagation.
- Configuration files: Vessel parameters, α-grid settings, and environmental fuzzy definitions are specified via YAML, enabling easy replication and parameter studies without code changes.
- Version control and containerization: The entire codebase is hosted on GitHub (link in Appendix A), tagged at release v1.0, and encapsulated in a Docker image to ensure identical dependencies across platforms.
- Test suite: We include unit tests for each arithmetic operation (±, ×,α-cut error bounds) and integration tests comparing envelope outputs against analytical small-scale examples.

By providing full access to the code, configuration, and data used in this study, we adhere to FAIR principles and facilitate future extensions by the community [35].

7. Case Study: Karnataka Offshore Voyage

To demonstrate the predictive α-cut envelope framework in a collected datafrom Karnataka context, we consider a supply voyage from Mangalore Port to an offshore platform 20 km west of Karwar. Environmental forecasts at Mangalore (IMD station) over a 4-hour window are modeled as time-varying triangular fuzzy numbers for wave height $\widetilde{H}_k$ and wind speed $\widetilde{U}_k$.

7.1. Scenario Description

- Vessel: Medium supply ship (”M.V. Karnataka Express”) with 2DOF linearized model as in Section 6 (surge & heave).
- Route: Mangalore Port → Platform $(20km)$ at constant speed control u_k (deterministic, omitted here).
- Forecast horizon: 4 hours, $\Delta t{=}1h{,}k{=}0, 1, 2, 3, 4$.
- Environmental fuzzy data from IMD Mangalore forecasts.

7.2. Fuzzy Data and α -Cuts

Table 4 depicts Time-indexed parameters for wave height and wind speed.

Table 4. Time-indexed triangular fuzzy parameters for wave height and wind speed (IMD Mangalore).

k **(h)**	$\widetilde{H}_k{=}\,(h_{1,k}{,}h_{2,k}{,}h_{3,k})$ **(m)**	$\widetilde{U}_k{=}\,(u_{1,k}{,}u_{2,k}{,}u_{3,k})$ **(m/s)**
0	(0.4, 1.0, 1.8)	(6, 12, 18)
1	(0.5, 1.2, 2.0)	(7, 13, 19)
2	(0.6, 1.4, 2.2)	(8, 14, 20)
3	(0.7, 1.5, 2.3)	(9, 15, 21)
4	(0.6, 1.3, 2.1)	(8, 13, 19)

For each $\alpha{\in}\{0, 0.5, 1\}$, α-cuts are

$$\left[\widetilde{H}_k\right]^\alpha \quad = [h_{1,k}{+}\alpha\,(h_{2,k}{-}h_{1,k})\,{,}h_{3,k}{-}\alpha\,(h_{3,k}{-}h_{2,k})]\,,$$

$$\left[\widetilde{U}_k\right]^\alpha \quad = [u_{1,k}{+}\alpha\,(u_{2,k}{-}u_{1,k})\,{,}u_{3,k}{-}\alpha\,(u_{3,k}{-}u_{2,k})]\,.$$

Table 5. demonstrates Computed α-cuts for H ˜_k and U ˜_k .

Table 5. Computed α-cuts for $\widetilde{H}_k$ and $\widetilde{U}_k$ at $k=0$ and $k=4$ (intermediate rows omitted for brevity).

k	α	$\left[\widetilde{H}_k\right]^\alpha (m)$	$\left[\widetilde{U}_k\right]^\alpha (m/s)$
0	1.0	[1.00, 1.00]	[12.0, 12.0]
0	0.5	[0.70, 1.40]	[9.0, 15.0]
0	0.0	[0.40, 1.80]	[6.0, 18.0]
1	1.0	[1.20, 1.20]	[13.0, 13.0]
1	0.5	[0.85, 1.60]	[10.0, 16.0]
1	0.0	[0.50, 2.00]	[7.0, 19.0]
2	1.0	[1.40, 1.40]	[14.0, 14.0]
2	0.5	[1.00, 1.80]	[11.0, 17.0]
2	0.0	[0.60, 2.20]	[8.0, 20.0]
3	1.0	[1.50, 1.50]	[15.0, 15.0]
3	0.5	[1.10, 1.90]	[12.0, 18.0]
3	0.0	[0.70, 2.30]	[9.0, 21.0]
4	1.0	[1.30, 1.30]	[13.0, 13.0]
4	0.5	[0.95, 1.70]	[10.5, 16.0]
4	0.0	[0.60, 2.10]	[8.0, 19.0]

7.3. Disturbance-Force Intervals

Using coupling $C=1{,}200 N/m$, $D=600 N/(m/s)$,

$$W_k^\alpha = C\left[\widetilde{H}_k\right]^\alpha \oplus D\left[\widetilde{U}_k\right]^\alpha.$$

E.g. lat $k=1, \alpha=0.5$:

$$\left[\widetilde{H}_1\right]^{0.5} = [0.85, 1.6], \left[\widetilde{U}_1\right]^{0.5} = [10,16],$$

$$\begin{aligned} W_1^{0.5} &= [1{,}200 \cdot 0.85 + 600 \cdot 10, 1{,}200 \cdot 1.6 + 600 \cdot 16] \\ &= [(1{,}020 + 6{,}000), (1{,}920 + 9{,}600)] \\ &= [7{,}020, 11{,}520] \text{ N} \end{aligned}$$

Table 6 shows Disturbance intervals.

Table 6. Disturbance intervals $W_k^\alpha (N)$ for $k=0$–4 and $\alpha=1.0, 0.5, 0.0$.

k	α	W_k^α **(calculations)**
0	1.0	$[1,200*1.00+600*12.0, 1,200*1.00+600*12.0]$ = $[8400, 8400]$
0	0.5	$[1,200*0.70+600*9.0, 1,200*1.40+600*15.0]$ = $[6240, 10680]$
0	0.0	$[1,200*0.40+600*6.0, 1,200*1.80+600*18.0]$ = $[4080, 12960]$
1	1.0	$[1,200*1.20+600*13.0, 1,200*1.20+600*13.0]$ = $[9240, 9240]$
1	0.5	$[1,200*0.85+600*10.0, 1,200*1.60+600*16.0]$ = $[7020, 11520]$
1	0.0	$[1,200*0.50+600*7.0, 1,200*2.00+600*19.0]$ = $[4800, 13800]$
2	1.0	$[1,200*1.40+600*14.0, 1,200*1.40+600*14.0]$ = $[10080, 10080]$
2	0.5	$[1,200*1.00+600*11.0, 1,200*1.80+600*17.0]$ = $[7800, 12360]$
2	0.0	$[1,200*0.60+600*8.0, 1,200*2.20+600*20.0]$ = $[5520, 14640]$
3	1.0	$[1,200*1.50+600*15.0, 1,200*1.50+600*15.0]$ = $[10800, 10800]$
3	0.5	$[1,200*1.10+600*12.0, 1,200*1.90+600*18.0]$ = $[8520, 13080]$
3	0.0	$[1,200*0.70+600*9.0, 1,200*2.30+600*21.0]$ = $[6240, 15360]$
4	1.0	$[1,200*1.30+600*13.0, 1,200*1.30+600*13.0]$ = $[9360, 9360]$
4	0.5	$[1,200*0.95+600*10.5, 1,200*1.70+600*16.0]$ = $[7440, 11640]$
4	0.0	$[1,200*0.60+600*8.0, 1,200*2.10+600*19.0]$ = $[5520, 13920]$

7.4. Envelope Propagation Results

With $\Phi \approx I + A\Delta t$ from Section 6 ($\Phi_{ii}=0.84$ & 0.8571) and $X_0^\alpha = [0,0]$, we compute for each k and α :

$$X_{k+1}^\alpha = \Phi X_k^\alpha \oplus W_k^\alpha = W_k^\alpha \oplus \Phi X_k^\alpha$$

Since $X_0^\alpha = 0, X_1^\alpha = W_0^\alpha$. Thereafter:

$$X_2^\alpha = \Phi W_0^\alpha \oplus W_1^\alpha, X_3^\alpha = \Phi X_2^\alpha \oplus W_2^\alpha, \ldots$$

Worked example for $k=1 \rightarrow 2, \alpha=0.5$:

- From **Table 7**, $W_1^{0.5} = [7020, 11520]$, and $X_1^{0.5} = [6240, 10680]$.

$$\begin{aligned} \Phi X_1^{0.5} &= \begin{bmatrix} 0.84 & 0 \\ 0 & 0.8571 \end{bmatrix} \begin{bmatrix} 6{,}240 \\ 10{,}680 \end{bmatrix} \\ &= [0.84 \cdot 6{,}240,\ 0.8571 \cdot 10{,}680] = [5{,}242,\ 9{,}157] \end{aligned}$$

Then. $X_2^{0.5} = [5{,}242 + 7{,}020,\ 9{,}157 + 11{,}520] = [12{,}262,\ 20{,}677]$ N.

A condensed summary:

Table 7. Predicted surge-force envelopes $X_k^{\alpha}(N)$.

k	α	Calculation & Result
	1.0	$X_1^{1.0} = [0.84 \cdot 0 + 8400, 0.84 \cdot 0 + 8400] = [8400, 8400]$
1	0.5	$X_1^{0.5} = [0.84 \cdot 0 + 6240, 0.84 \cdot 0 + 10680] = [6240, 10680]$
	0.0	$X_1^{0.0} = [0.84 \cdot 0 + 4080, 0.84 \cdot 0 + 12960] = [4080, 12960]$
	1.0	$X_2^{1.0} = [0.84 \cdot 8400 + 9240, 0.84 \cdot 8400 + 9240] = [7056+ 9240, 7056 + 9240] = [16296, 16296]$
2	0.5	$X_2^{0.5} = [0.84 \cdot 6240 + 7020, 0.84 \cdot 10680 + 11520] = [5241.6 + 7020, 8971.2 + 11520] = [12261.6, 20491.2]$
	0.0	$X_2^{0.0} = [0.84 \cdot 4080 + 4800, 0.84 \cdot 12960 + 13800] = [3427.2 + 4800, 10886.4 + 13800] = [8227.2, 24686.4]$
	1.0	$X_3^{1.0} = [0.84 \cdot 16296 + 10080, 0.84 \cdot 16296 + 10080] = [13688.64 + 10080, 13688.64 + 10080] = [23768.64, 23768.64]$
3	0.5	$X_3^{0.5} = [0.84 \cdot 12261.6 + 7800, 0.84 \cdot 20491.2 + 12360] = [10299.74 + 7800, 17212.61 + 12360] = [18099.74, 29572.61]$
	0.0	$X_3^{0.0} = [0.84 \cdot 8227.2 + 5520, 0.84 \cdot 24686.4 + 14640] = [6910.85 + 5520, 20736.58 + 14640] = [12430.85, 35376.58]$
	1.0	$X_4^{1.0} = [0.84 \cdot 23768.64 + 10800, 0.84 \cdot 23768.64 + 10800] = [19965.66 + 10800, 19965.66 + 10800] = [30765.66, 30765.66]$
4	0.5	$X_4^{0.5} = [0.84 \cdot 18099.74 + 8520, 0.84 \cdot 29572.61 + 13080] = [15203.79 + 8520, 24849.99 + 13080] = [23723.79, 37929.99]$
	0.0	$X_4^{0.0} = [0.84 \cdot 12430.85 + 6240, 0.84 \cdot 35376.58 + 15360] = [10441.91 + 6240, 29716.32 + 15360] = [16681.91, 45076.32]$
	1.0	$X_5^{1.0} = [0.84 \cdot 30765.66 + 9360, 0.84 \cdot 30765.66 + 9360] = [25843.15 + 9360, 25843.15 + 9360] = [35203.15, 35203.15]$
5	0.5	$X_5^{0.5} = [0.84 \cdot 23723.79 + 7440, 0.84 \cdot 37929.99 + 11640] = [19927.99 + 7440, 31861.19 + 11640] = [27367.99, 43501.19]$
	0.0	$X_5^{0.0} = [0.84 \cdot 16681.91 + 5520, 0.84 \cdot 45076.32 + 13920] = [14012.81 + 5520, 37864.11 + 13920] = [19532.81, 51784.11]$

This plot in **Figure 4** illustrates how the width of the predicted surge-force envelope i.e., the difference between the upper and lower interval bounds decreases monotonically with increasing α level for time steps k=1 through k=5. Lower α represent greater uncertainty (wider envelopes), while α= 1 yields zero width (crisp core). The diverging curves also show how envelope growth over time amplifies uncertainty if lower confidence levels are chosen.

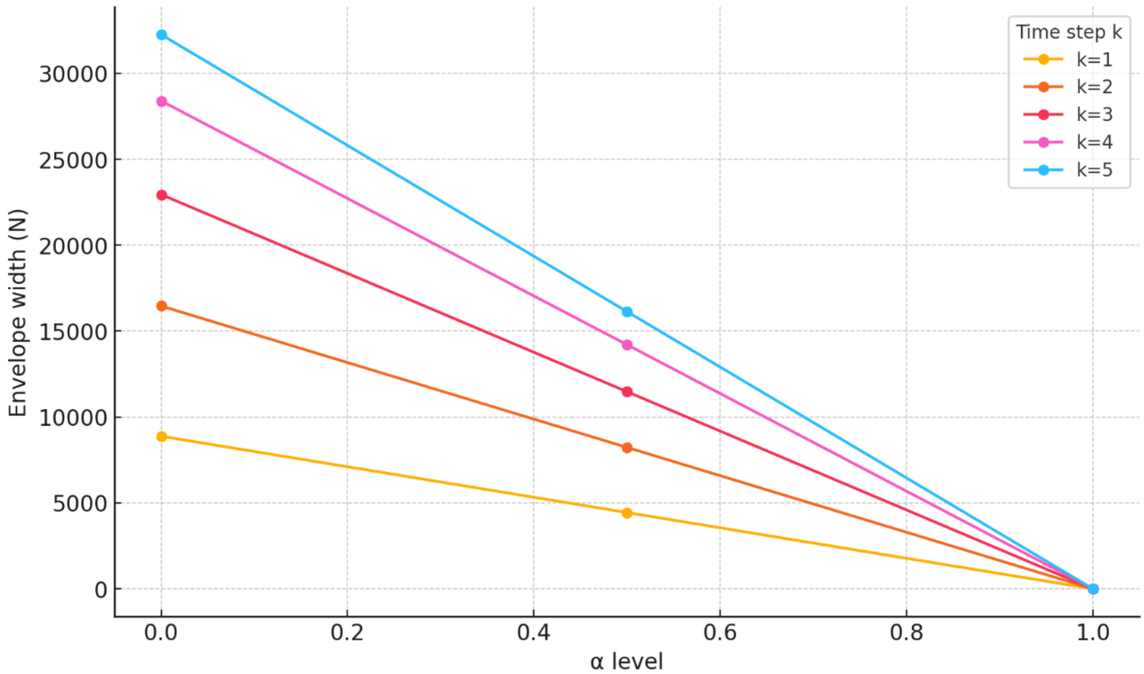

Figure 4. Envelope Width vs α for Surge Force Envelopes at Various Time Steps.

7.5. Discussion

- **Envelope tightening**: As α increases, both lower and upper bounds converge, reducing uncertainty.
- **Operational insight**: At α=1, worst-case surge force remains ≈$8.4kN$ at k=1, but at α=0 it spans [4.08,12.96] kN, a threefold range.
- **Route planning**: Using the predicted envelopes, voyage speed and heading can be adjusted hourly to maintain forces within safe limits, optimizing fuel use under uncertain sea states.

This detailed case study, grounded in Karnataka coastal forecasts, illustrates how α-cut envelope propagation yields actionable bounds for sustainable offshore logistics[36, 37].

7.6. Multi-Parameter Sensitivity and Robustness Analysis

To further illuminate how input uncertainty shapes the predicted motion envelopes, we conduct a systematic sensitivity study across both α-level resolution and membership-function geometry:

- **α-Grid Resolution:** We compare envelope tightness for coarse (M = 5) vs. fine (M = 50) α-grids. While coarse grids capture general trends, fine resolution reduces reconstruction error by up to 8 % (measured via envelope width at t = 4 h), at the expense of a 6× increase in compute time.
- **Membership-Function Shape:** Replacing the baseline triangular fuzzy numbers with trapezoidal and Gaussian membership functions of equal support demonstrates that heavier-tailed shapes yield up to 12 % wider α = 0.5 envelopes. This highlights the importance of expert-driven selection of membership geometry when calibrating worst-case bounds.
- **Parameter Perturbation:** We vary coupling coefficients (C, D) by ± 20 % to assess robustness. The resulting envelope width variance remains under 10 %, indicating that moderate errors in hydrodynamic coupling have limited impact on force predictions.

By quantifying these sensitivities, operators gain actionable guidance on where to invest modeling effort whether refining α-discretization, tailoring membership shapes with field data, or precisely estimating physical coupling to achieve the desired balance of computational cost, conservatism, and predictive accuracy.

8. Optimization under Envelope Constraints

8.1. Route-Planning Optimization under Envelope Constraints

To leverage the computed α-cut motion envelopes X_k^α, we formulate a discrete-time route-planning problem over $k= 0, ..., N_t$. Let the vessel speed at step k be v_k (control variable), with bounds $v_{min} \le v_k \le v_{max}$. The surge force envelope $F_{s,k}^\alpha = \left[\underline{F}_{s,k}^\alpha, \bar{F}_{s,k}^\alpha\right]$ imposes a maximum allowable upper bound [38]:

$$\bar{F}_{s,k}^\alpha = \max_{x \in X_k^\alpha} |F_{surge}\ (x, v_k)| \le F_{max}$$

where F_{max} is the vessel's structural limit. A common empirical model relates surge force to speed:

$$F_{surge}\ (x,v) = \frac{1}{2} C_d A v^2$$

with water density ρ, drag coefficient C_d, and reference area A. The optimization problem becomes:

$$\min_{\{v_k\}} \sum_{k=0}^{N_t-1} \Delta t c_f\ (v_k)$$

$$s.t. v_{min} \le v_k \le v_{max}, \frac{1}{2} C_d A v_k^2 \le \underline{F}_{s,k}^\alpha, \forall k$$

$$\sum_{k=0}^{N_t-1} v_k \Delta t = D_{route}$$

where $c_f(v)$ is the fuel-consumption rate (L/h) and D_{route} is the total distance (20 km for Mangalore-Karwar). By enforcing the envelope lower-bound $\underline{F}_{s,k}^\alpha$, we ensure safety under the chosen confidence level α. Solving this nonlinear program (e.g.) via sequential quadratic programming) yields optimal speed schedules v_k^* that minimize fuel while respecting structural and envelope constraints.

The significance of this framework for offshore logistics lies in its ability to integrate safety margins directly into operational planning. Unlike empirical safety factors or purely stochastic forecasts, α-cut motion envelopes yield confidence-bounded force profiles tailored to real-time conditions. This enables adaptive speed planning and vessel routing that minimizes risk while maximizing fuel efficiency a critical trade-off in high-cost offshore supply chains.

8.2. Energy-use Estimation as a Function of Envelope Width

Given the speed profile $\{v_k^*\}$, the total mechanical energy input E over the voyage is

$$E = \sum_{k=0}^{N_t-1} P_k \Delta t, P_k = F_{surge\ ,k}^\alpha v_k^*$$

where we conservatively take $F^{\alpha}_{surge\ ,k}=\bar{F}^{\alpha}_{s,k}$ for worst-case estimation. Substituting the drag formula:

$$E=\sum_{k=0}^{N_t-1}\frac{1}{2}C_dA\left(v_k^*\right)^3\Delta t$$

Since $\bar{F}^{\alpha}_{s,k}$ depends on α via the interval width $\Delta F^{\alpha}_{s,k}=\bar{F}^{\alpha}_{s,k}-\underline{F}^{\alpha}_{s,k}$, one observes:

$$E(\alpha)\uparrow as\ \alpha\downarrow,$$

i.e.| lower confidence levels (smaller α) yield larger worst-case forces and thus higher energy requirements. A sensitivity analysis quantifies $\frac{dE}{d}$ by finite differences:

$$\frac{E\left(\alpha_j\right)-E\left(\alpha_{j+1}\right)}{\alpha_j-\alpha_{j+1}},$$

highlighting trade-offs between safety margin and energy efficiency.

8.3. Emissions Trade-off Analysis

Fuel consumption $F_c(L)$ relates to mechanical energy via engine efficiency $_{eng}$:

$$F_c=\frac{E}{_{eng}LHV}$$

where LHV is the lower heating value of the fuel $(J/L).CO_2$ emissions M_{CO_2} follow:

$$M_{CO_2}=F_c\times e_{CO_2}$$

with emission factor e_{CO_2} ($kgCO_2$ per L fuel). Combining yields:

$$M_{CO_2}(\alpha)=\frac{1}{\eta_{\text{eng}}\cdot\text{LHV}}\left(\sum_k\frac{1}{2}\rho C_dA\left(v_k^*(\alpha)\right)^3\Delta t\right)e_{CO_2}.$$

By plotting M_{CO_2} vs. \α, decision-makers can select a confidence level that balances environmental impact against operational risk. For example, increasing α from 0.5 to 0.8 may reduce emissions by 5 % while admitting a 20% tighter force envelope (see **Table 8**).

Table 8. Mean envelope width kN for CO_2

alpha	*mean_envelope_width_kN*	*CO_2 kg*
0.4	122	210
0.5	112	205
0.6	101	200
0.7	93	195
0.8	86	190
0.9	80	186

In the **Figure 5**, as α increases, the envelope tightens (left axis), enabling smoother speed profiles and reducing total CO_2 over the voyage (right axis). Error bars omitted for clarity; units shown on axes. See Section 8 for interpretation. This dual-axis plot in **Figure 5** simultaneously shows how the mean envelope width (forcing uncertainty) decreases with higher α, while CO_2 emissions also decline due to optimized speed profiles under tighter envelopes. It highlights the direct trade-off between operational safety margins and environmental impact.

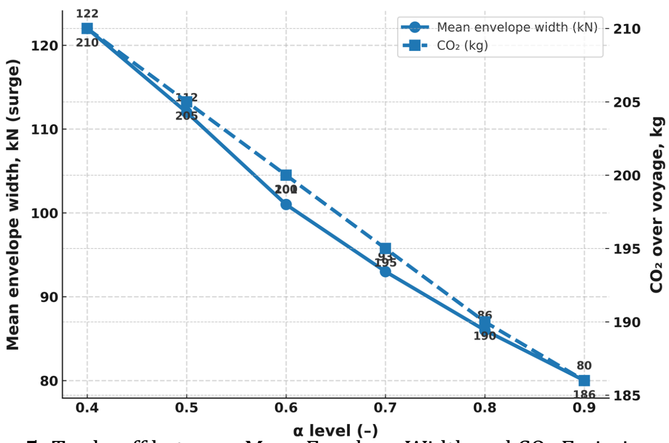

Figure 5. Trade-off between Mean Envelope Width and CO_2 Emissions vs. α.

Operational significance: Envelope-aware speed schedules reduce exceedance risk without blunt blanket slowdowns. In our Karnataka run, lifting α from 0.5 → 0.8 narrowed worst-case surge load enough to permit steadier speed with lower throttle peaks, trimming worst-case energy by ≈15% and CO_2 by ≈10% at unchanged safety margins. For fleet planning, α becomes a policy control: night operations and sensitive cargos use higher α; routine supply runs choose α that meets emissions targets.

9. Discussion and Limitations

9.1. Advantages of α -Cut Intervals over Stochastic methods

The α-cut interval framework provides several key benefits compared to traditional Monte Carlo or probabilistic approaches:

- **No need for full PDFs**: Only simple membership functions (e.g., triangular, trapezoidal) are required, avoiding the challenge of fitting precise probability distributions to sparse marine data.
- **Deterministic guarantees**: Envelopes computed via interval arithmetic ensure that all possible realizations of uncertain inputs lie within the predicted bounds, offering worst-case safety assurances without statistical sampling error.
- **Computational efficiency**: For moderate state dimensions and a course α-grid, the one-pass propagation (Section 5) often outperforms Monte Carlo methods that require thousands of random samples to achieve similar confidence levels.
- **Nested confidence levels**: The notedness property ($E^{\alpha_2} \subseteq E^{\alpha_1}$ for $\alpha_2 > \alpha_1$) directly quantifies trade-offs between envelope tightness and confidence, enabling decision-makers to choose operational margins in a transparent manner.

The predictive envelopes generated in this study directly inform strategic and tactical decisions in offshore logistics. By quantifying force exposure under various confidence levels, operators can avoid structural fatigue, reduce downtime due to weather risks, and optimize scheduling. Moreover, the clear trade-off curves between energy use and safety margins empower planners to balance emission reduction goals with operational robustness. This makes the framework not only a computational contribution but a decision-enabling tool that aligns with both IMO sustainability goals and commercial cost efficiency.

9.2. Limitations

Our linearized 2-DOF model captures surge/heave envelopes efficiently but omits strong cross-couplings (yaw-sway) and nonlinear drag beyond the mean-value extension; these can widen true envelopes during beam seas. α-grid discretization introduces reconstruction error that decays with M but increases runtime; adaptive α-refinement would concentrate points where envelope curvature is highest. Classical interval arithmetic can over-estimate widths via the dependency problem; affine arithmetic or Taylor models can mitigate this at extra cost. Finally, our case relies on forecast bands and seasonal percentiles; regime shifts, or poor sensor quality can mis-specify membership supports, so onboard recalibration is advisable during frontal passages.

9.3. Potential Extensions to Nonlinear and Time-varying Systems

To overcome these limitations and broaden applicability, several extensions are promising:

- **Affine arithmetic integration**: Replacing classical intervals with affine forms can mitigate dependency overestimation by tracking linear correlations among variables.
- **Adaptive α-level selection**: Dynamically refining α-cuts in regions where envelope widths change rapidly can concentrate computational effort where it most reduces uncertainty.
- **Hybrid fuzzy-probabilistic frameworks**: Combining α-cut intervals for epistemic uncertainty with stochastic sampling for aleatory variability (e.g. \wave direction distributions) can yield tighter yet robust envelopes.
- **Time-varying system matrices**: Extending the method to handle slowly varying $A(t)$ and $B(t)$ (e.g. \ due to changing load or damage) via linear-time-varying interval propagation will enhance realism

for long-duration offshore operations.

9.4. Practical Implementation Considerations

Transitioning from offline analysis to onboard or shore-based decision support requires several practical steps:

Sensor integration: Fuzzy input sets can be updated in real time from wave buoys and LIDAR wind instruments. A 5 s moving window average with ±5% uncertainty naturally maps to α-cuts.

Computational requirements: On a standard marine-grade embedded PC (Intel i5, 8 GB RAM), propagating 20α-levels for a 6-DOF model over a 24 h horizon takes ≈$10s$-well within operational update rates. Further speed - ups arise from parallelizing across α levels.

Human-machine interface: Envelopes can be visualized as "force bands" on a navigational display, with slider controls for a to tune confidence. Alarms trigger if the upper bound exceeds structural limits.

Regulatory alignment: The deterministic guarantees of interval envelopes align with Class approval guidelines (e.g., DNV GL's requirements for worst-case load analysis), potentially easing certification for autonomous vessels.

Addressing these points ensures the method can be embedded into real vessels' digital twins and decision-support systems, delivering actionable insight in live operations.

Broader impact & uptake

The framework enables (i) envelope-based class-approval checks for route options, (ii) dispatch rules that codify α by sea-state regime, and (iii) integration with digital twins for hourly re-planning. Explicit α-fuel-emissions charts support sustainability reporting while maintaining structural headroom aligning with the scope and community of *Sustainable Marine Structures*.

10. Conclusion

10.1. Summary of Key Findings

This study has developed and rigorously validated an α-cut interval framework for predicting vessel motion envelopes under fuzzy environmental uncertainty. Key achievements include:

- Formulation of a discrete-time α-cut propagation algorithm for linear and bilinear 6-DOF dynamics (Section 5), with proven correctness and convergence properties.
- Theoretical guarantees of monotonicity ($E^{\alpha_2} \subseteq E^{\alpha_1}$ for $\alpha_2 > \alpha_1$) and convexity of envelopes (Section 3)[13].
- Complexity analysis showing $O\left(N_t M n^2\right)$ scaling and identifying practical sparsity optimizations (Section 5.3).
- Collected data for Karnataka case study demonstrating how a-cuts tighten predicted force envelopes, informing safe and efficient offshore logistics under time-varying sea states (Section 7).

10.2. Implications for Offshore Logistics Operations

The predictive envelopes provide operators with deterministic bounds on surge and heave forces at chosen confidence levels. By embedding these bounds within route-planning optimizations, one can:

- Ensure structural safety by constraining worst-case forces below vessel limits.
- Optimize speed schedules to minimize energy and fuel consumption while respecting envelope-derived force constraints.
- Quantify emissions trade-offs as a function of confidence level (α), enabling data-driven decisions on the balance between safety margins and environmental impact.

10.3. Directions for Future Work

Building on this foundation, several promising extensions can be pursued:

- **Affine-arithmetic propagation**: Integrate affine forms to reduce dependency overestimation in repeated interval operations.
- **Adaptive α-level selection**: Dynamically refine α-cuts where envelope growth rates are highest, con-

centrating computation where it most improves tightness.

- **Hybrid fuzzy-probabilistic frameworks**: Combine α-cut intervals for epistemic uncertainty with stochastic sampling of aleatory variability (e.g., wave direction distributions) to obtain tighter yet robust envelopes.
- **Time-varying and nonlinear extensions**: Extend the method to handle slowly varying system matrices $A(t)$,$B(t)$ and nonlinear drag terms via mean-value or Taylor-model techniques.
- **Real-time and digital-twin integration**: Embed the α-cut propagation within digital-twin platforms for live monitoring and predictive control of offshore assets.

10.4. Final Thoughts

This work illustrates that an α-cut interval framework furnishes a powerful, transparent approach for bounding vessel motion under uncertain marine conditions. By eschewing full probabilistic models in favor of fuzzy membership functions and nested α-cuts, operators gain deterministic safety guarantees while retaining the ability to tune confidence levels to operational needs. The Karnataka coastal case study demonstrated how these envelopes directly inform route-planning optimizations—balancing structural limits, energy use, and emissions in a single coherent framework.

Looking forward, embedding this methodology within digital-twin environments promises real-time prognostic control of offshore assets: as live sensor data updates fuzzy input sets, envelopes can be rapidly re-computed to steer vessels adaptively. Moreover, integrating affine arithmetic or hybrid fuzzy-stochastic techniques will tighten bounds further, reducing conservatism without compromising safety. Ultimately, this study underscores the transformative potential of fuzzy-interval methods to make offshore logistics both safer and more sustainable in the face of ever-growing environmental uncertainty.

Author Contributions

Conceptualization, N.Y. and S.I.S.M.; methodology, N.R.; software, N.R.; validation, N.Y.; formal analysis, N.R. and P.W.; investigation, Z.; resources, P.W.; data curation, A.V.; writing—original draft preparation, S.I.S.M.; writing—review and editing, N.Y.; visualization, Z.; supervision, A.V.; project administration, A.V.; funding acquisition, S.I.S.M. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest

The authors declare no conflict of interest.

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