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ARTICLE

Modeling of Waves on a Dual-Porous Pre-Stressed Transversely Isotropic Impedance and Irregular Boundary Medium with Thermal and Dual Pore Sources

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ABSTRACT

Occurrences in the forms of vibrational phenomena have both positive and negative impacts on the Earth's crust and materials. Scientists in the field of seismology and emerging technologies often hinge their innovations and applications on the nature of material compositions. Owing to this, we present in this work a surface wave solution that results from a dual-porous pre-stressed transversely isotropic impedance medium with an irregular boundary under heat stress based on Green-Lindsay thermoelasticity, and derived through the principles of mathematical analysis associated with wave motion. The irregularity of the boundary is assumed to be in the form of a corrugated surface. This is represented as a trigonometric Fourier series in which the wave number and the amplitude associated with the corrugated surface of the medium affect the motion of the wave. Moreover, initial stress and dual porosity sources are incorporated into the modeled problem to enrich its physical composition. Due to the satisfaction of the adopted displacement components within the classical wave equation, we employed the harmonic solution method to find the analytical solution and perform analysis on the modeled equations of motion. Following this, we derived the fundamental analytical solution for the various distributions of double porosities, thermal flux, shear and normal stresses, and displacement components of the surface wave on the transversely isotropic material. We demonstrate the dependency of the wave propagation on these interacting physical quantities including dual porosities, initial stress and the grooved boundary surface parameters such as the wave number and amplitude of material's corrugation. Thus, it suffices to state that researchers in geophysics, material sciences, and mechatronics applications, among others, would find this model useful.

Keywords: Transversely Isotropic Material; Grooved Boundary; Impedance; Initial Stress; Thermal Stress; Double-Porosity

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1. Introduction

The investigation of seismic wave propagation and modulation caused by natural occurrences such as earthquake is crucial for understanding the Earth's interior and other seismic happenings around different surfaces of the Earth's crust. The Earth, by its compositions, is structured in different layers. Nevertheless, these layers are subjected to seismic waves resulting from earthquake activity. Thus, the propagation properties of these seismic waves in layered structures are significantly of essence as a result of the applications primarily related to geophysics; in relation with prospecting for minerals, mechanical and structural engineering, among others. Generally speaking, the Earth's crust as a material medium, is not homogeneous. Different forms of heterogeneous characteristics abounds. The developments in heterogeneous characteristics occurred due to varying elastic properties wuth depth. The heterogeneous occurrences could also be as a result of the phenomena involving develop, growth, and coalescence of micro-cracks within the solid rock materials. Subsequently, substantial evidence exists in the literatures that rock structures in the Earth's crust contains many forms of heterogeneities, which can be represented mathematically using several functions. This postulations gave rise to the idea that a lot of studies which were already carried out could depict the impacts of various forms of heterogeneity through linear, quadratic, cubic, quartic or even exponential, or other nonlinear functions of depth on the SH-waves propagation, in particular.

Normally, earthquake phenomena are triggered as a result of body forces, and are formulated as space-time dependent impulsive line sources, usually expressed by the Dirac delta function whose resulting elastic displacement in layered structures is determined by making use of the Green's function technique. Green's function technique has demonstrated to be a proven mathematical approach for finding the solution of problems in elastodynamics associated with solid mechanics of materials involving impulsive sources. For instance, various studies have examined the propagation behavior of Love-type of waves by utilizing Green's function method to handle impulsive point source problem in a heterogeneous medium, viscoelastic functionally graded half-space media respectively. Hence, this analysis is similar to investigations on the effect of point source on SH-wave propagation through orthotropic crustal layers, as widely documented in the literatures. The notion of Love waves are such that they are surface waves that propagate through an elastic layer of short thickness resting against an elastic half-space in welded contact. Love waves are polarized horizontal shear waves whose vibrations of its particle are parallel to the horizontal plane of the layer and normal to the direction of propagation of the wave. Researchers have observed that the devastating structural collapse during earthquakes are often due to SH-motion, closely related to Love-type waves. Owing to their high propagation speed among various surface waves, their applications span across seismology, non-destructive testing of layered media, and examination of laminated and coated materials.

In addition, researchers in this field of mechanics devote great attention to studies involving initial stress on material exhibiting disturbances while incorporating thermal field, magnetic field and maybe porosities, etc., under various thermal relaxation time parameters. Their relevance in exploring and interpreting information about seismic waves, earthquakes, volcanoes and mechanical acoustics occurrences are paramount. Thus, studies which are concerned with the classical theory of thermal elasticity suggests that if an elastic solid is put under a thermal loading, the effect is immediately observed at a distance from the source. This implies that thermal waves propagate at infinite speed, resulting in unrealistic physical predictions. In contrast, non-classical thermoelastic emerged in the late 20th century, incorporating a flux-rate term into Fourier's law of heat conduction. This gave rise to a generalized theory of thermo-elasticity that yields a hyperbolic heat transport equation that admits finite speeds for thermal signals. In line with this similar idea, Green and Lindsay opined a temperature-rate term into the constitutive variables characterizing a medium, and formulated a temperature-rate-dependent thermoelasticity theory. The theory does not distort the classical Fourier's law of heat conduction, especially for bodies with a center of symmetry. They showed that this theory also predicts a finite speed for heat wave propagation. All of these developments are widely recognized and referenced in the literature.

In spite of this, the analysis of stress-strain factors in materials, which plays a significant role in understanding the strength and fatigue characterizations of materials, is of crucial importance to engineering destructive testing and a range of other industrial applications in solid mechanics of materials such as in geophysics and seismology. Such materials could be orthotropic, monoclinic, isotropic, or transversely isotropic, etc. However, they are best classified as isotropic and anisotropic materials as in Spencer [1] and Abd-Alla et al. [2]. Transversely isotropic materials characterizations are often symmetric about an axis and postulated to be normal to a plane of isotropy. For instance, unidirectional fibre composite lamina whose fibres are circular in cross-section, geological layers of rocks are all equally interpreted or understood as transversely isotropic. Hence, describing a model or developing formulations that provide insight into vibratory modulations and analysis on transversely isotropic medium becomes imperative for researchers and scientists. These are achieved through mathematical and theoretical approach or even experimental studies. Mathematical models capable of capturing wave solutions in transversely isotropic materials have been proposed by Chadwick et al. [3], Lalawmpuia et al. [4,5], Gupta et al. [6], Ding et al. [7] and Chattopadhyay [8]. The results obtained by these authors were based on the specific material characterizations considered in their respective studies.

Furthermore, incorporating some of these environmental factors or the internal properties of the medium such as initial stress, voids or porosity, etc., is not uncommon in model formulations and analysis of physical materials. Authors in the literature, including Anya et al. [9,10], Othman et al. [11], Bayones et al. [12], Singh et al. [13], Acharya et al. [14], Kundu et al. [15], Zhu et al. [16] and Dhua et al. [17], made contributions along similar lines. Their studies focused on the investigation and enumeration of the influences of initial stress, homogeneity, and heterogeneity for the propagation of waves in a transversely isotropic medium under thermal flux, and also on the dynamics for a rotating grooved and impedance boundary anisotropic material which characterizes their respective examinations. Some of the authors in their examinations were also of the view that the propagation of waves is impacted by the temperature effect and the presence of voids in the material. Porous and multiporous media are defined as materials possessing one or more types of pore parameters or inclusions in their structure. This means that a porous medium also called poroelastic medium is a type of material medium made up of the solid skeleton including with pore spaces that is often filled with fluid. Voids or pores on materials are one important aspect of material characterization encountered in almost every substance on the planet and surely this is without thermo-elastic effects. Moreover, researches have shown that heating of a material containing pores tend to expand the pores ration on the material and the reverse is the case when the material is cooled. That is, when the material cools down, the pores components on the material contract. This entails that particle migration back to their original shape also plays a vital role in the contraction and expansion of the voids on the material. Waves modulation and propagation through single, dual or multiple porous medium is of great essence in many engineering applications such as the oil and gas industry, geotechnical and geophysics engineering, chemical engineering, amongst others. Following the problem models associated with elastic stability for anisotropic material, examinations involving fluid saturated porous layer has been discovered to be very useful in theories and applications. It is evident that the layers of the Earth's crust comprises of all kinds of rocks; limestone, shale, etc., demonstrating poroelasticity in nature. And thus, during the events of earthquake dynamics, these poroelastic rocks are subjected to seismic waves propagation. This made it possible for scientists, geoscientists and engineers, to devote great interests for seismic waves propagation in poroelastic materials. The following authors in the literatures Cowin et al. [18] and Nunziato et al. [19] made formulations and proposed both the linear and non-linear theories for wave propagation in isotropic elastic media with voids. Their proposition and development of linear and nonlinear theories on elastic material with voids infused nothing of mechanical or energetic importance, however, finite deformations are still possible during stress loadings. Cowin and Nuziato theories for voids in elastic material were carried out or proposed because the idea is to factor in bulk density of the medium into two fields: voids volume fractional fields and the density field of matrix material. Such factorization of bulk density infused a nondependent kinematic parameter or variable. This is called the change in volume fractional field during the stress-strain loading process on the material medium.

In a different vein, material surfaces and boundaries are often of distinct shapes, either planar or nonplanar. In this work, we utilized the concept of nonplanar or irregular surface boundary conditions likened to corrugation or a grooved surface condition of the material. Corrugation refers to the ridges and grooves present on a material, whether artificially or naturally designed, and their effects on elastic wave propagation are significant. Several impacts such as dispersion of the wave, phase velocity and even show of direction to wave propagation in a bunch of materials are highly envisaged and alterations feasible. As a matter of fact, interface grooves decreases phase velocity in Lamb waves while surface grooves increases phase velocity in Lamb waves. It is also worthy to note that grooves are crucial in creating guided waves such as the spin waves in magnetic materials or even in exploiting or manipulating the properties of surface waves and in particular; Rayleigh waves. Hence, optical waveguides, design of smart structures, magnonics field of study leading to development of new materials and nondestructive evaluation of materials are some specific instances through which grooves on material could play importance to our everyday material productions and usage. Asano [19] investigated reflection and refraction across interfaces using a grooved boundary conditions represented by a periodic trigonometric series. Other relevant works include Maleki et al. [20], Chowdhury et al. [21], Lalvohbika et al. [22], Prikazchikov et al. [23] and Rogerson et al. [24], who studied isotropic and incompressible transversely isotropic materials with corrugated surface initial stress. Also, authors in the literatures have, nevertheless, made contributions to these developments of double porous and corrugated surface studies along interfaces using several physical material compositions, as the case may be, and as posited by the following authors; Mishra et al. [25] whose works were conducted on the transmission of Love waves due to an impulsive line source in a heterogeneous double porous rock structure, Dutta et al. [26] works centered on finding the solution of waves in a nonlocal fiber-reinforced double porous material structure under fractional-order heat and mass transfer, Panji et al. [27] investigated on a half-space TD-BEM model for a seismic corrugated orthotropic stratum and Dai [28] work dwelt on providing solutions for Love waves in double porous media while elastic wave propagation and attenuation in a double-porosity dual-permeability medium were investigated by Berryman et al. [29].

In addition, impedance boundary conditions are widely employed by researchers. Impedance on materials in a mechanical wave propagation, entails the measure of a material's resistance to the flow of matter or energy carried along by a wave. More so, it actually stipulates the ratio of the pressure or force moving the wave to the yielding velocity. Studies involving impedance gives a soothe understanding especially in analysis regarding wave characteristics across interfaces of different layered materials where reflection and refraction takes place. Matching impedance and impedance mismatch are two important concepts in wave propagation on impedance materials. If the impedance of two different media matches then the wave energy is with less reflection and with more transmission. And when the case for a mismatch impedance occurs on two different materials some of the wave energy reflects and the others are transmitted. The quantity transmitted and reflected depends on the magnitude of the difference in impedance of the two media. This connote that from a physical point of reference, so far as there is no variation in the impedance, propagation of the wave is not affected or impacted. This is also helpful in the design of mechanical-acoustic media, design of musical instruments. In geophysics and geotechnical, impedance is applied in analyzing seismic waves and in understanding the structure of the Earth's surfaces and subsurface.

It is also very crucial to engineering metamaterials designs which invariably have artificially engineered characteristics and capability to control wave propagation, Anya et al. [10]. Consequently, all these makes it possible for Scientists to employ suitable boundary surfaces for a particular examination on wave propagations.

Based on the literatures reviewed above, the investigation aims to model and present an analytical solution for surface wave modulation on an initially stressed, corrugated-impedance boundary of a homogeneous dual porous transversely isotropic material with dual pore sources and thermal loadings. The thermal loading is considered to take precedents from Green-Lindsay thermo-elastic theory where the two thermal relaxation times parameters are incorporated in the heat conduction equations and the stress equations. Thus, this infuses a bridge in gaps in the literatures, and most worthy to note; as it stipulates or proposes the analytical solutions of such models not found in combined kind as presented in this investigation. For instance, Singh et al. [30] model wholly dealt on plane harmonic waves in a thermo-elastic solid with double porosity and without the considerations of initial stress, impedance and grooved surfaces of the their considered materials. Thus, it is important to examine the nature of our present work/model or study as regards to the considered physical material structure and its characterization arising from solid mechanics since material characterization and formulations are ever evolving for technological advancements. This research work undoubtedly leads to seismology and thus, seismic solutions of wave characterizations and distributions and in particular surface waves vis-à-vis Love, Stoneley and Rayleigh types of waves for the dual transversely isotropic considered material and its composition. These kinds of waves on dual porous transversely isotropic medium are of great essence in studies involving mechatronics devices, biomedical magnetic imaging associated with noninvasive medical imaging tests that produces good account of almost every internal structures in the human body including blood vessels, bones, organs, etc. using magnet and wave principles. Subsequently, this research work employs a wave dynamic procedure called the harmonic method in its investigation and the solution of the problem presented in a two dimensional geometry of the material. Consequently upon this, the developed analytical results of the thermal field distribution, the stresses on the material, the dual porous distribution and the displacement distributions, on the surface of the material are derived and presented. We analytically demonstrate the dependency of the fields' distributions and by extension the wave propagation on these interacting physical quantities including initial stress, grooved-impedance parameters and the two times thermal relaxation constants.

2. Materials and Methods

Mathematical Formulation of the Problem

The constitutive equation for the stress-strain relation that depicts a grooved-impedance surface of a transversely isotropic material considering initial stress, Anya et al. [31] and double porosity, Singh et al. [30], on the material with thermal stress, Abd-Alla et al. [2], and Anya et al. [31], compositions is thus, presented:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{12} & 0 & 0 & 0\\ 0 & C_{22} & C_{23} & 0 & 0 & 0\\ 0 & 0 & C_{22} & 0 & 0 & 0\\ 0 & 0 & 0 & 1/2(C_{22}-C_{23}) & 0 & 0\\ 0 & 0 & 0 & 0 & C_{66} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix}-$$

$$\begin{bmatrix}\beta_1\left(1+\nu_o\dfrac{\partial}{\partial t}\right)(T-T_o)\\ \beta_1\left(1+\nu_o\dfrac{\partial}{\partial t}\right)(T-T_o)\\ \beta_{ij}\left(1+\nu_o\dfrac{\partial}{\partial t}\right)(T-T_o)\\ 0\\ 0\\ 0\end{bmatrix}+\begin{bmatrix}(\chi_1\phi+\chi_2\vartheta)\\ (\chi_1\phi+\chi_2\vartheta)\\ (\chi_1\phi+\chi_2\vartheta)\\ 0\\ 0\\ 0\end{bmatrix}-P\begin{bmatrix}(\delta_{11}+\varpi_{11})\\ (\delta_{22}+\varpi_{22})\\ (\delta_{33}+\varpi_{33})\\ (\delta_{23}+\varpi_{23})\\ (\delta_{13}+\varpi_{13})\\ (\delta_{12}+\varpi_{12})\end{bmatrix}. \quad (1)$$

The parameters in Equation (1) i.e., C_{11}, C_{12}, C_{22}, C_{23}, and C_{66} denotes the material constants, $\varepsilon_{ij}=(u_{i,j}+u_{j,i})/2$, entails the strain tensor, σ_{ij} represents the stress tensor which gives the field distribution of the stress on the material, whereas δ_{ij} is defined to be the Kronecker–delta function such that $i=j=1,2,3..\ \varpi_{ij}=(u_{i,j}-u_{j,i})/2$, represents the rigid body rotation tensor, P denotes initial stress parameter on the transversely isotropic material. u_i denotes the components of the displacements of the wave on the material. β_{ij} denote thermal moduli of the grooved-impedance transversely isotropic material, ν_o denotes one of the thermal relaxation times parameters, while T denote the temperature of the material. ϕ and ϑ are field components of the material due to double porosities while χ_1, χ_2 represents parameters due to double porosities of the transversely isotropic material. Thus, the governing balance laws under the influence of Green and Lindsay (G-L) theory of thermo-elasticity considering the stress-strain relations characterized by grooved-impedance surface of a transversely isotropic material under initial stress and double porosities on the material follows:

$$\sigma_{ij,j} = \rho \ddot{u}_i \tag{2}$$

$$\xi_1(\phi_{,ii}) + \xi_2(\vartheta_{,ii}) - \chi_1(u_{i,i}) - \omega_o\phi - \varpi\vartheta + \gamma_4 T = \gamma_1\ddot{\phi}, \tag{3}$$

$$\xi_2(\phi_{,ii}) + \xi(\vartheta_{,ii}) - \chi_2(u_{i,i}) - \varpi\phi - \gamma_3\vartheta + \gamma_5 T = \gamma_2\ddot{\vartheta}, \tag{4}$$

$$\begin{aligned}\frac{\partial}{\partial x_i}(\kappa_{ij}\frac{\partial T}{\partial x_j}) = \rho c_v(\frac{\partial}{\partial t} + \tau_o\frac{\partial^2}{\partial t^2})T + T_o\gamma_4(\frac{\partial}{\partial t} + \tau_o\frac{\partial^2}{\partial t^2})\phi + \\ T_o\gamma_5(\frac{\partial}{\partial t} + \tau_o\frac{\partial^2}{\partial t^2})\vartheta + T_o\beta_{ij}(\frac{\partial}{\partial t})\varepsilon_{ij}.\end{aligned} \tag{5}$$

The terms ν_o and τ_o in the field equations above, are the known two term Green-Lindsay thermal relaxation constants or parameters which satisfies the hypothesis $\nu_o \geq \tau_0 \geq 0$. More so, when we consider $\tau_o > 0$, then $\nu_o > 0$, thus this entails Equation (5) predicts a short speed of modulation of thermal signals and if $\nu_o = \tau_0 = 0$, it stipulates that Equations (1) and (5) gives coupled thermo-elasticity theory with dual porosity effects. T is the temperature of the medium. $T - T_o$ is the difference in temperature of the medium and reference temperature of the medium. In addition, the assumption $|T - T_o| << T_0$ is used to replace $T - T_o$ with T in Equation (1). Also, ρ represents the density of transversely isotropic material, while $\gamma_i, i = 1,2,3,4,5$; represents parameters due to double porosities on the grooved-impedance transversely isotropic material, κ_{ij} represent conductivity tensor, c_v is the specific heat at constant deformation. Owing to this Mathematical formulations and the posited governing laws stated above, our Mathematical analysis to this model problem would be centered in the directions of x_1 x_2-plane of coordinates such that the direction of x_3 = 0 and the displacements $u_1 \neq u_2 \neq 0$ are however, coupled. Nevertheless, Equations (2)–(5) results to the components of the dynamic equations which are as follows:

$$\begin{aligned}C_{11}u_{1,11} + C_{12}u_{2,21} + B_1u_{1,22} + B_3u_{2,12} = \rho\ddot{u}_1 + \\ \beta_1\left(1 + \nu_o\frac{\partial}{\partial t}\right)(T - T_o) - (\chi_1\phi + \chi_2\vartheta)_{,1},\end{aligned} \tag{6}$$

$$\begin{aligned}C_{22}u_{2,22} + B_1u_{2,11} + B_3u_{1,21} = \rho\ddot{u}_2 + \beta_1\left(1 + \nu_o\frac{\partial}{\partial t}\right) \\ (T - T_o) - (\chi_1\phi + \chi_2\vartheta)_{,2},\end{aligned} \tag{7}$$

$$\xi_1(\phi_{,ii}) + \xi_2(\vartheta_{,ii}) - \chi_1(u_{i,i}) - \omega_o\phi - \varpi\vartheta + \gamma_4 T = \gamma_1\ddot{\phi}, \tag{8}$$

$$\xi_2(\phi_{,ii}) + \xi(\vartheta_{,ii}) - \chi_2(u_{i,i}) - \varpi\phi - \gamma_3\vartheta + \gamma_5 T = \gamma_2\ddot{\vartheta}, \tag{9}$$

$$\begin{aligned}\frac{\partial}{\partial x_i}(\kappa_{ij}\frac{\partial T}{\partial x_j}) = \rho c_v(\frac{\partial}{\partial t} + \tau_o\frac{\partial^2}{\partial t^2})T + T_o\gamma_4(\frac{\partial}{\partial t} + \tau_o\frac{\partial^2}{\partial t^2})\phi + \\ T_o\gamma_5(\frac{\partial}{\partial t} + \tau_o\frac{\partial^2}{\partial t^2})\vartheta + T_o\beta_{ij}(\frac{\partial}{\partial t})(u_{1,1} + u_{2,2}).\end{aligned} \tag{10}$$

Where; $B_1 = (C_{66} - P/2)$, $B_1 = (B_3 + C_{12})$, $B_3 = (C_{66} + P/2)$.

3. Solution of the Problem

Here, we consider the characterization of the dual porous, corrugated-impedance surface of a homogeneous transversely isotopic half-space under heat flux, and thus, employ the normal mode approach for wave analysis, such that the wave displacements, dual porous field effects and thermal assumptions take the form below:

$$\begin{aligned}(u_j, \phi, \vartheta, T - T_0 = \Phi) = \{\bar{u}_j(x_2), \bar{\phi}(x_2), \bar{\vartheta}(x_2), \\ \bar{\Phi}(x_2)\}e^{\omega t + ibx_1}, j = 1,2.\end{aligned} \tag{11}$$

By employing Equation (11) into Eqs. (6–10), we have the equations below:

$$(B_1D^2 - C_{11}b^2 - \rho\omega^2)\bar{u}_1 + (iB_2bD)\bar{u}_2 + (\chi_1 ib)\bar{\phi} + (\chi_2 ib)\bar{\vartheta} + bis\bar{\Phi} = 0, \tag{12}$$

$$(iB_3bD)\bar{u}_1 + (C_{22}D^2 - B_1b^2 - \rho\omega^2)\bar{u}_2 + (\chi_1 D)\bar{\phi} + (\chi_2 D)\bar{\vartheta} + sD\bar{\Phi} = 0, \tag{13}$$

$$-\chi_1 ib\bar{u}_1 - \chi_1 D\bar{u}_2 + (-b^2\xi_1 + \xi_1 D^2 - \omega_o - \gamma_1\omega^2)\phi + (-b^2\xi_2 + \xi_2 D^2 - \varpi)\vartheta + \gamma_4\Phi = 0, \tag{14}$$

$$-\chi_2 ib\bar{u}_1 - \chi_2 D\bar{u}_2 + (-b^2\xi_2 + \xi_2 D^2 - \varpi)\bar{\phi} + (-b^2\xi + \xi D^2 - \gamma_2\omega^2 - \gamma_3)\bar{\vartheta} + \gamma_5\Phi = 0, \tag{15}$$

$$-T_0\beta_1 bi\omega\bar{u}_1 - T_0\beta_1\omega D\bar{u}_2 - \omega T_0\gamma_4(1+\tau_0\omega)\bar{\phi} - \omega T_0\gamma_5(1+\tau_0\omega)\bar{\vartheta} + (\kappa D^2 - \omega\rho c_v(1+\tau_0\omega) - \kappa b^2)\bar{\Phi} = 0. \tag{16}$$

Here, $D = d/dx_2$, is an ordinary differential operator as per convention. Consequently, non-trivial solution of Equations (12)–(16), that is, for $(\bar{u}_1, \bar{u}_2, \bar{\phi}, \bar{\vartheta}, \bar{\Phi}) \neq 0$, the determinant of the system of Equations (12)–(16), yields the characteristic polynomial equation below:

$$(C_1D^{10} + C_2D^8 + C_3D^6 + C_4D^4 + C_5D^2 + C_6)(\bar{u}_1, \bar{u}_2, \bar{\phi}, \bar{\vartheta}, \bar{\Phi}) = 0 . \tag{17}$$

where $s = -(1 + \nu_o\omega)$ and C_i, i =1, 2, 3, 4, 5, 6 (see **Appendix A**) showcase that the complex coefficients of the material parameters are eminent. Since ν_i, i =1, 2, 3, 4, 5 gives positive real roots of auxiliary Equation (17), we have the solutions in the form below whilst utilizing the harmonic method associated with wave analysis as presented below:

$$(\bar{u}_1, \bar{u}_2, \phi, \vartheta, \bar{\Phi}) = \sum_{n=1}^{5} (M_n, M_{1n}, M_{2n}, M_{3n}, M_{4n}) e^{-\nu_n x_2}, \tag{18}$$

Here, M_n, M_{1n}, M_{2n}, M_{3n} and M_{4n} are functions wholly dependent on the wave number b as linked to the grooved boundary surface and the frequency ω which is complex in nature, and in the direction of the $x_1$coordinate of the surface wave modulation. Thus, this entails that using Equation (18) into Equations (12)–(16), yields the following relations below:

$$M_{1n} = H_{1n}M_n, \; M_{2n} = H_{2n}M_n, \; M_{3n} = H_{3n}M_n, \; M_{4n} = H_{4n}M_n. \tag{19}$$

Where: $H_{1n} = [(B_1\nu_n^2 - C_{11}b^2 - \rho\omega^2) + B_3b^2]\nu_n / [(B_2\nu_n^2 - (C_{22}\nu_n^2 - B_1b^2 - \rho\omega^2))bi]$,

$H_{2n} = \{(\chi_1 ib - P_1 - P_4 - P_7Q_0) / (P_0 - P_3 + P_6 + P_7Q_2)\} + \{(P_2 + P_5 - P_7Q_2 - \chi_1\nu_n) / (P_0 - P_3 + P_6 + P_7Q_2)\}H_{1n}$,

$H_{3n} = \{Q_0 + H_{1n}Q_1 + H_{2n}Q_2\}, P_0 = \xi_1\nu_n^2 - \xi_1 b^2 - \omega_0 - \gamma_1\omega^2$,

$H_{4n} = \{T_0\beta_1\omega(bi - \nu_n H_{1n}) + \omega T_0(1+\tau_0\omega)(\gamma_4 H_{2n} + \gamma_5 H_{3n})\} / (\kappa\nu_n^2 - C_v\rho\omega(1+\tau_0\omega) - \kappa b^2)$,

$$P_1 = \frac{(\xi_2(\nu_n^2 - b^2) - \bar{\omega})\{(-\gamma_5\beta_1T_0\omega i) + (\chi_2 ib)(\kappa\nu_n^2 - \rho C_v\omega(1+\tau_0\omega))\}}{\{(\xi(\nu_n^2 - b^2) - \gamma_2 - \gamma_3)(\kappa\nu_n^2 - \rho C_v\omega(1+\tau_0\omega)) + (\gamma_5^2T_0\omega)(1+\tau_0\omega)\}},$$

$$P_2 = \frac{(\xi_2(\nu_n^2 - b^2) - \bar{\omega})\{(\gamma_5\beta_1T_0\nu_n) - (\chi_2\nu_n)(\kappa\nu_n^2 - \rho C_v\omega(1+\tau_0\omega))\}}{\{(\xi(\nu_n^2 - b^2) - \gamma_2 - \gamma_3)(\kappa\nu_n^2 - \rho C_v\omega(1+\tau_0\omega)) + (\gamma_5^2T_0\omega)(1+\tau_0\omega)\}},$$

$$P_3 = \frac{(\xi_2(\nu_n^2 - b^2) - \bar{\omega})\{(\kappa\nu_n^2 - \rho C_v\omega(1+\tau_0\omega))(\xi_2(\nu_n^2 - b^2) - \bar{\omega}) - T_0\omega\gamma_5\gamma_4(1+\tau_0\omega)\}}{\{(\xi(\nu_n^2 - b^2) - \gamma_2 - \gamma_3)(\kappa\nu_n^2 - \rho C_v\omega(1+\tau_0\omega)) + (\gamma_5^2T_0\omega)(1+\tau_0\omega)\}},$$

$$P_4 = \frac{T_0\omega\gamma_4 b\beta_1 i}{(\kappa\nu_n^2 - \rho C_v\omega(1+\tau_0\omega))}, P_5 = \frac{T_0\omega\gamma_4\beta_1\nu_n}{(\kappa\nu_n^2 - \rho C_v\omega(1+\tau_0\omega))},$$

$$P_6 = \frac{T_0\omega\gamma_4^2\omega(1+\tau_0\omega)}{(\kappa\nu_n^2 - \rho C_v\omega(1+\tau_0\omega))}, P_7 = \frac{T_0\omega\gamma_4\gamma_5\omega(1+\tau_0\omega)}{(\kappa\nu_n^2 - \rho C_v\omega(1+\tau_0\omega))},$$

$$Q_0 = \frac{\{(\chi_2 ib)(\kappa\nu_n^2 - \rho C_v\omega(1+\tau_0\omega)) - (\gamma_5\beta_1T_0\omega i)\}}{\{(\xi(\nu_n^2 - b^2) - \gamma_2 - \gamma_3)(\kappa\nu_n^2 - \rho C_v\omega(1+\tau_0\omega)) + (\gamma_5^2T_0\omega)(1+\tau_0\omega)\}},$$

$$Q_1 = \frac{\{(\gamma_5\beta_1T_0\nu_n) - (\chi_2\nu_n)(\kappa\nu_n^2 - \rho C_v\omega(1+\tau_0\omega))\}}{\{(\xi(\nu_n^2 - b^2) - \gamma_2 - \gamma_3)(\kappa\nu_n^2 - \rho C_v\omega(1+\tau_0\omega)) + (\gamma_5^2T_0\omega)(1+\tau_0\omega)\}},$$

$$Q_2 = \frac{\{(\kappa\nu_n^2 - \rho C_v\omega(1+\tau_0\omega))(\xi_2(\nu_n^2 - b^2) - \bar{\omega}) - (T_0\omega\gamma_5\gamma_4)(1+\tau_0\omega)\}}{\{(\xi(\nu_n^2 - b^2) - \gamma_2 - \gamma_3)(\kappa\nu_n^2 - \rho C_v\omega(1+\tau_0\omega)) + (\gamma_5^2T_0\omega)(1+\tau_0\omega)\}},$$

4. Boundary Conditions of the Material

We make a supposition such that the equation of impedance-grooved surface of the transversely isotropic half-space with dual pores be denoted as $x_2 = \eta(x_1)$. Here, $\eta(x_1)$ is considered a periodic function and certainly independent of x_3 coordinate. Thus, we can assume a trigonometric Fourier series of $\eta(x_1)$ by following Asano [32]:

$$\eta(x_1) = \sum_{l=1}^{\infty} (\eta_l e^{ilbx_1} + \eta_{-l} e^{-ilbx_1}), \tag{20}$$

where $\eta_l$and η_{-l} are Fourier expansion coefficients and l, is the series expansion order. Let us initiate the constants a, R_l and I_l as follows: $\eta_1^{\pm} = a/2$, $\eta_l^{\pm} = (F_l + I_l)/2, l = 2,3...$, and$\eta(x_1)$ = $a\cos bx_1$ + $F_2\cos 2bx_1$ + $I_2\sin 2bx_1$ +... + $F_l\cos lbx_1$ + $I_l\sin lbx$. F_l and I_l are the Fourier cosine and sine Fourier coefficients respectively. It suffices that the nature of the grooved or corrugated boundary surface can be denoted with the help of cosine terms, that is, by taking $\eta(x_1) = a\cos bx_1$, . Where a is the amplitude of the grooved boundary, and b is the wavenumber associated with the grooved boundary with $2\pi/b$ as the wavelength.

i. Stress w.r.t x_2 are continuous , i.e.

$$\sigma_{22} - \eta'(x_1)\sigma_{21} + \omega Z_2 u_2 + P = 0\,, \tag{21}$$

$$\sigma_{12} - \eta'(x_1)\sigma_{11} + \omega Z_1 u_1 + \mathrm{P}\varpi_{12} = 0\,, \tag{22}$$

at $x_2 = \eta(x_1)$, for all x_1 and t.

ii. the double porous Type-I and Type-II boundary conditions takes the form:

$$\xi_1\phi_{,2} + \xi_2\vartheta_{,2} = P_1 e^{\omega t + ibx}, \tag{23}$$

$$\xi_2\phi_{,2} + \xi\vartheta_{,2} = P_2 e^{\omega t + ibx}, \tag{24}$$

at $x_2 = \eta(x_1)$, respectively.

iii. thermal boundary conditions

$$T_{,2} + hT = 0. \tag{25}$$

Here, Z_1 and Z_2 entails the impedance parameters of the grooved-impedance transversely isotropic material. Also, $h \to 0$, gives thermally insulated boundary characteristics of the material, and $h \to \infty$, gives isothermal boundary conditions of the material. While P_1 and P_2 represents the dual porous source parameters on the material.

5. Results

In this section, we present the case where application of the impedance and grooved boundary conditions with dual porous source parameters, considering G-L thermoelasticity of the transversely isotropic medium, results to the system of equations below whilst using the fact that $h \to 0$, i.e., for thermally insulated boundary:

$$\begin{aligned}&\{ab\sin bx_1((B_1 ibH_{1n} - B_3\nu_n)) - (C_{22}\nu_n H_{1n} + \beta_1 H_{2n}(1+\nu_o\omega))\\&+(\chi_1 H_{2n} + \chi_2 H_{3n}) + \omega Z_2 H_{1n}\} M_n e^{-\nu_n \xi(x_1)} = 0,\end{aligned} \tag{26}$$

$$\begin{aligned}&\{ab\sin bx_1(C_{11}ib - C_{12}\nu_n H_{1n} + (\chi_1 H_{2n} + \chi_2 H_{3n}) -\\&\beta_1 H_{4n}(1+\nu_o\omega)) + \{(B_3 ibH_{1n} \ -\nu_n B_1)\} + \omega Z_1\}\\&M_n e^{-\nu_n\xi(x_1)+\omega t + ibx_1} = Pab\sin bx_1\,,\end{aligned} \tag{27}$$

$$(\xi_1\nu_n H_{2n} + \xi_2\nu_n H_{3n}) M_n e^{-\nu_n\xi(x_1)} = -P_1 \tag{28}$$

$$(\xi_2\nu_n H_{2n} + \xi\nu_n H_{3n}) M_n e^{-\nu_n\xi(x_1)} = -P_2 \tag{29}$$

$$\{(-\nu_n\} H_{4n} M_n e^{-\nu_n\xi(x_1)} = 0,\ n = 1,2,3,4,5. \tag{30}$$

The analytic solution of the field distributions of stresses, thermal and displacement components and dual porous distributions are finally derived for $M_n, n = 1,2,3,4,5,$ in Equations (26)–(30). These are easily obtained using any symbolic solver like Mathematica Software, etc. Hence, the distribution of the components of displacements of the surface wave, double porous and thermal field distributions, and the normal and shear stress quantities, for the considered material, yields the following:

$$u_1 = M_n e^{-\nu_n x_2 + \omega t + ibx_1},\ u_2 = M_n H_{1n} e^{-\nu_n x_2 + \omega t + ibx_1},$$

$$\phi = M_n H_{2n} e^{-\nu_n x_2 + \omega t + ibx_1},\ \vartheta = M_n H_{3n} e^{-\nu_n x_2 + \omega t + ibx_1},$$

$$T = M_n H_{4n} e^{-\nu_n x_2 + \omega t + ibx_1} + T_0,$$

$$\begin{aligned}\sigma_{22} = \{&-(C_{22}\nu_n H_{1n} + \beta_1 H_{4n}(1+\nu_o\omega)) +\\&(\chi_1 H_{2n} + \chi_2 H_{3n})\} M_n e^{-\nu_n x_2 + \omega t + ibx_1} - P,\end{aligned}$$

$$\begin{aligned}\sigma_{11} = \ \{&C_{11}ib - C_{12}\nu_n H_{1n} + (\chi_1 H_{2n} + \chi_2 H_{3n}) -\\&\beta_1 H_{4n}(1+\nu_o\omega)\} M_n e^{-\nu_n x_2 + \omega t + ibx_1} - P,\end{aligned}$$

$$\sigma_{12} = \ \{(B_3 ibH_{1n} \ -\nu_n B_1)\} M_n e^{-\nu_n x_2 + \omega t + ibx_1},$$

$$\sigma_{21} = \ \{(B_1 ibH_{1n} - B_3\nu_n)\} M_n e^{-\nu_n x_2 + \omega t + ibx_1},\ n = 1,2,3,4,5.$$

And thus, we can physically and mathematically assert to the dependency of the considered parameters of initial stress, grooved and impendence parameters as linked with the amplitude and wave number of the corrugation, dual pore sources and thermal times constants on the fields' distributions as evident in the $M_n, n = 1,2,3,4,5$ viz a viz M_{1n}, M_{2n}, M_{3n} and M_{4n}.

6. Conclusions

This work deals with the closed-form or analytical solution of plane surface waves propagating on the grooved-impedance boundary of a dual-porous prestressed transversely isotropic material with thermal stress, based on the Green-Lindsay theory, in which two thermal relaxation times parameters are incorporated in the heat and stress equations, respectively. Aside from the formulation of the model, closed-form relations of the thermal distribution, dual porous and stresses distributions, and the displacement components of the wave motion on the grooved-impedance transversely isotropic material were derived. It is found that these physical parameters of corrugation, impedance, double pore sources, and the heat flux influence the modulation of the wave fields of displacements, thermal fields and stress distributions on the material. These were achieved by the utilization of the harmonic solution method and the employment of a non-trivial solution to the reduced associated equations of motion, which resulted in a polynomial equation of degree 10; however, with 5 positive complex roots characterizing the fields' distributions. Hence, a robust numerical approach may be needed in computing these roots of the polynomial for graphical visualizations of the resulting derived closed-form solutions of the fields distributions.

In addition, we deduced that for negligible prestressed occurrences and dual pores source parameters on the considered material, the determinant of Equations (26)–(30), for $M_n \neq 0; n = 1,2,3,4,5$ would give the dispersion relation of Rayleigh-type of wave as a particular case of the model. Also, other forms of surface waves like the Stoneley waves (which are a generalized kind of Rayleigh waves in which we assume that waves are propagated along the common boundary of two different semi-infinite media) and Love waves can equally be deduced following their individual boundary conditions they are associated with.

Thus, it is crucial to note that this research output serves as a basis for examining information in the areas of destructive testing in materials, material sciences and designs, seismology and geophysics; where grooved-impedance characterizations and dual porosity in materials are of paramount importance. Such specifics are linked to mechanical systems, in which impedance applies to the analysis of vibrating structures and musical equipment. Often, these surface waves are applied in other areas, such as in biomedical devices like biosensors used in biology to analyze deoxyribonucleic acid properties, virus detection, and so on. Also, the measurements of toxic material substances in Medicine and Chemistry can be achieved by employing some of these surface waves in laboratory settings.

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Conflicts of Interest

The author declare no conflict of interest.

References

[1] Spencer, A.J.M., 1972. Deformations of Fibre-Reinforced Materials; Oxford University Press: Oxford, UK.

[2] Abd-Alla, A.M., Abo-Dahab, S.M., Khan, A., 2017. Rotational effects on magneto- thermoelastic stonely, Love, and Rayleigh waves in fibre-reinforced anisotropic general viscoelastic media of higher order. CMC. 53, 49–72. DOI: https://doi.org/10.3970/cmc.2017.053.052

[3] Chadwick, P., Seet L.T.C., 1970. Wave propagation in a transversely isotropic heat con-ducting elastic material. Mathematica. 17, 255–274. DOI: https://doi.org/10.1112/S002557930000293X

[4] Lalawmpuia T., Singh S.S., 2020. Effect of initial stresses on the elastic waves in transversely isotropic thermoelastic materials. Engineering Reports. 2(1), e12104. DOI: https://doi.org/10.1002/eng2.12104

[5] Lalawmpuia, T., Singh, S.S., Zorammuana, C., et al., 2022. A mathematical model of shear wave propagation in the incompressible transversely isotropic thermoelastic half-spaces. International Journal of Engineering, Science and Technology. 14(1), 52–59. DOI: https://doi.org/10.4314/ijest.v14i1.5

[6] Gupta, R.R., 2014. Surface wave characteristics in a micropolar transversely isotropic half-space underlying an inviscid liquid layer. International Journal of Applied Mechanics. 19, 49–60.

[7] Ding, H., Chen, W., Zhang, L., 2006. Elasticity of Transversely Isotropic Materials. Springer Science & Business Media: New York, NY, USA. DOI: https://doi.org/10.1007/1-4020-4034-2

[8] Chattopadhyay, A., 1978. On the strong SH motion in a transversely isotropic layer lying over an isotropic elastic material due to a momentary point source. Indian Journal of Pure and Applied Mathematics. 9, 886–892.

[9] Anya, A.I., Akhtar, M.W., Abo-Dahab, M.S., et al., 2018. Effects of a magnetic field and initial stress on reflection of SV-waves at a free surface with voids under gravity. Journal of the Mechanical Behavior of Materials. 27(5–6). DOI: https://doi.org/10.1515/jmbm-2018-0002

[10] Anya, A.I., Kaneez, H., 2024. Inclined Loading and Thermal Field Dynamics in a Rotating Periodic

Grooved-impedance Boundary Anisotropic Material. Jokull Journal. 74(11), Available form: https://www.jokulljournal.com/coredoux/index.php/archive/part/74/11/1/?currentVol=74&currentissue=11

[11] Othman, M.I.A., Abo-Dahab, S.M., Alosaimi, H.A., 2018. Effect of inclined load and magnetic field in a micropolar thermoelastic medium possessing cubic symmetry in the context of G-N theory, Multidiscipline Modeling in Materials and Structures. 14(2), 306–321. DOI: https://doi.org/10.1108/MMMS-08-2017-0086

[12] Bayones, F.S., Hussein, N.S., Abd-Alla, A.M., et al., 2021. The influence of heterogeneity and initial stress on the propagation of Love-type wave in a transversely isotropic layer subjected to rotation. Science Progress. 104(3). DOI: https://doi.org/10.1177/00368504211041496

[13] Singh, A.K., Kumari, N., Chattopadhyay, A., et al., 2016. Smooth moving punch in an initially stressed transversely isotropic magnetoelastic medium due to shear wave. Mech Adv Mater Struct. 23, 774–783.

[14] Acharya, D.P., Roy, I., Sengupta, S., 2009. Effect of magnetic field and initial stress on the propagation of interface waves in transversely isotropic perfectly conducting media. Acta Mechanica. 202, 35–45.

[15] Kundu, S., Gupta, S, Manna, S., 2014. SH-type waves dispersion in an isotropic medium sandwiched between an initially stressed orthotropic and heterogeneous semi-infinite media. Meccanica. 49, 749–758.

[16] Zhu, H., Zhang, L., Han J., et al., 2014. Love wave in an isotropic homogeneous elastic half-space with a functionally graded cap layer. Applied Mathematics and Computation. 231, 93–99. DOI: https://doi.org/10.1016/j.amc.2013.12.167

[17] Dhua, S., Chattopadhyay A., 2015. Torsional wave in an initially stressed layer lying between two inhomogeneous media. Meccanica. 50, 1775–1789.

[18] Cowin, S.C., Nunziato, J.W., 1983. Linear elastic materials with voids. Journal Elasticity. 13, 125–147. DOI: https://doi.org/10.1007/BF00041230

[19] Nunziato, J.W., Cowin, S.C., 1979. A nonlinear theory of elastic materials with voids. Archive for Rational Mechanics and Analysis. 72(2), 175–201. DOI: https://doi.org/10.1007/BF00249363

[20] Maleki, F., Jafarzadeh, F., 2023. Model tests on determining the effect of various geometrical aspects on horizontal impedance function of surface footings. Scientia Iranica.

[21] Chowdhury, S., Kundu, S., Alam, P., et al., 2021. Dispersion of Stoneley waves through the irregular common interface of two hydrostatic stressed MTI media. Scientia Iranica. 28(2), 837–846.

[22] Lalvohbika, J., Singh, S.S., 2020. Waves due to corrugated interface in incompressible transversely isotropic fiber-reinforced elastic half-spaces. Mechanics of Advanced Materials and Structures. 29(12), 1720–1729. DOI: https://doi.org/10.1080/15376494.2020.1838005

[23] Prikazchikov, D.A., Rogerson G.A., 2004. On surface wave propagation in incompressible, transversely isotropic, pre-stressed elastic half-spaces. International Journal of Engineering Science. 42, 967–986. DOI: https://doi.org/10.1016/j.ijengsci.2003.10.003

[24] Rogerson, G.A., 1991. Some dynamic properties of incompressible, transversely isotropic elastic media. Acta Mechanica. 89, 179–186. DOI: https://doi.org/10.1007/BF01171254

[25] Mishra, A., Negi, A., 2004. Investigation on the transmission of Love waves due to an impulsive line source in a heterogeneous double porous rock structure. Scientific Reports. 14, 27499. DOI: https://doi.org/10.1038/s41598-024-77203-1

[26] Dutta, R., Das, S., Gupta, S., et al., 2023. Nonlocal fiber-reinforced double porous material structure under fractional-order heat and mass transfer. International Journal of Numerical Methods for Heat & Fluid. 33(11), 3608–3641. DOI: https://doi.org/10.1108/HFF-05-2023-0295

[27] Panji, M., 2023. A half-space TD-BEM model for a seismic corrugated orthotropic stratum. Engineering Analysis with Boundary Elements. 152, 655–677. DOI: https://doi.org/10.1016/j.enganabound.2023.04.032

[28] Dai, Z.J., Kuang, Z.-B., 2006. Love waves in double porosity media. Journal of Sound and Vibration. 296(4–5), 1000–1012. DOI: https://doi.org/10.1016/j.jsv.2006.03.029

[29] Berryman, J.G., Wang, H.F., 2000. Elastic wave propagation and attenuation in a double-porosity dual-permeability medium. International Journal of Rock Mechanics and Mining Sciences. 37(1–2), 63–78. DOI: https://doi.org/10.1016/S1365-1609(99)00092-1

[30] Singh, D., Kumar, D., Tomar, S.K., 2020. Plane harmonic waves in a thermoelastic solid with double porosity. Mathematics and Mechanics of Solids. 25(4), 869–886. DOI: https://doi.org/10.1177/1081286519890053

[31] Anya, A.I., John, E.E., 2025. Analytical solutions of waves in an initially stressed corrugated 2-D homogeneous transversely isotropic half-space with thermal and inclined mechanizations. Journal of Institutional Research, Big Data Analytics and Innovation. 1(2), 202–211.

[32] Asano, S., 1966. Reflection and refraction of elastic waves at a corrugated interface. Bulletin of the Seismological Society of America. 56(1), 201–221.

Appendix A

This appendix is provided to describe the various coefficients associated with Equation (17), i.e., the characteristic equation obtained from the non-trivial solution of the resulting Equations (12)–(16). These complex coefficients are essential for the evaluation of the characteristic roots needed for the formulation for the solutions of the field distributions given in Equation (11).

$$C_1=\kappa B_1 C_{22}\left(\xi\xi_1-\xi_2^2\right),$$

$$\begin{aligned}
C_2={}&(-\kappa\left(b^2i^2B_2B_3+K_7C_{22}\right)\left(\xi\xi_1-\xi_2^2\right)+B_1(-K_6\kappa\xi\xi_1+s\xi\omega T_0\beta_1\xi_1+K_6\kappa\xi_2^2-s\omega T_0\beta_1\xi_2^2-C_{22}(K_5\kappa\xi+\left(K_4\kappa+K_1\xi\right)\xi_1\\
&-2\kappa\varpi\xi_2-\left(K_1+2b^2\kappa\right)\xi_2^2)+\kappa\xi\chi_1^2-2\kappa\xi_2\chi_1\chi_2+\kappa\xi_1\chi_2^2)),
\end{aligned}$$

$$\begin{aligned}
C_3={}&(K_5K_7\kappa\xi C_{22}+K_6K_7\kappa\xi\xi_1+K_4K_7\kappa C_{22}\xi_1+K_1K_7\xi C_{22}\xi_1+biK_8s\xi C_{22}\xi_1-K_7s\xi\omega T_0\beta_1\xi_1-b^2i^2s\xi\omega B_3T_0\beta_1\xi_1\\
&-2K_7\kappa\varpi C_{22}\xi_2-K_6K_7\kappa\xi_2^2-K_1K_7C_{22}\xi_2^2-biK_8sC_{22}\xi_2^2-2b^2K_7\kappa C_{22}\xi_2^2+K_7s\omega T_0\beta_1\xi_2^2+b^2i^2s\omega B_3T_0\beta_1\xi_2^2\\
&-K_7\kappa\xi\chi_1^2-b^2i^2\kappa\xi B_3\chi_1^2+b^2i^2\kappa\xi C_{22}\chi_1^2+2K_7\kappa\xi_2\chi_1\chi_2+2b^2i^2\kappa B_3\xi_2\chi_1\chi_2-2b^2i^2\kappa C_{22}\xi_2\chi_1\chi_2-K_7\kappa\xi_1\chi_2^2-b^2i^2\kappa B_3\xi_1\chi_2^2\\
&+b^2i^2\kappa C_{22}\xi_1\chi_2^2+biB_2(-K_8s\xi\xi_1+K_8s\xi_2^2+biB_3(K_5\kappa\xi+\left(K_4\kappa+K_1\xi\right)\xi_1-2\kappa\varpi\xi_2-\left(K_1+2b^2\kappa\right)\xi_2^2)\\
&-bi\kappa\xi\chi_1^2+2bi\kappa\xi_2\chi_1\chi_2-bi\kappa\xi_1\chi_2^2)+B_1(K_5K_6\kappa\xi+K_4K_6\kappa\xi_1+K_1K_6\xi\xi_1-2K_6\kappa\varpi\xi_2-K_1K_6\xi_2^2-2b^2K_6\kappa\xi_2^2+\\
&C_{22}(K_4K_5\kappa+K_1K_5\xi-\kappa\varpi^2+\left(K_1K_4+K_2\gamma_5\right)\xi_1-2K_1\varpi\xi_2-2b^2\kappa\varpi\xi_2-K_3\gamma_5\xi_2-2b^2K_1\xi_2^2\\
&-b^4\kappa\xi_2^2+\gamma_4\left(K_3\xi-K_2\xi_2\right))+K_3s\xi\chi_1-K_2s\xi_2\chi_1-K_4\kappa\chi_1^2-K_1\xi\chi_1^2+K_2s\xi_1\chi_2-K_3s\xi_2\chi_2+2\kappa\varpi\chi_1\chi_2+2K_1\xi_2\chi_1\chi_2\\
&+2b^2\kappa\xi_2\chi_1\chi_2-K_5\kappa\chi_2^2-K_1\xi_1\chi_2^2-\omega T_0\beta_1(-2b^2s\xi_2^2+\xi\left(K_5s+\gamma_4\chi_1\right)-\xi_2\left(2s\varpi+\gamma_5\chi_1+\gamma_4\chi_2\right)+\xi_1\left(K_4s+\gamma_5\chi_2\right))))
\end{aligned}$$

$$\begin{aligned}
C_4={}&(-K_5K_6K_7\kappa\xi-K_4K_5K_7\kappa C_{22}-K_1K_5K_7\xi C_{22}-biK_5K_8s\xi C_{22}+K_7\kappa\varpi^2C_{22}+K_5K_7s\xi\omega T_0\beta_1+b^2i^2K_5s\xi\omega B_3T_0\beta_1\\
&-K_3K_7\xi C_{22}\gamma_4-K_4K_6K_7\kappa\xi_1-K_1K_6K_7\xi\xi_1-biK_6K_8s\xi\xi_1-K_1K_4K_7C_{22}\xi_1-biK_4K_8sC_{22}\xi_1+K_4K_7s\omega T_0\beta_1\xi_1\\
&+b^2i^2K_4s\omega B_3T_0\beta_1\xi_1-K_2K_7C_{22}\gamma_5\xi_1+2K_6K_7\kappa\varpi\xi_2+2K_1K_7\varpi C_{22}\xi_2+2biK_8s\varpi C_{22}\xi_2+2b^2K_7\kappa\varpi C_{22}\xi_2\\
&-2K_7s\varpi\omega T_0\beta_1\xi_2-2b^2i^2s\varpi\omega B_3T_0\beta_1\xi_2+K_2K_7C_{22}\gamma_4\xi_2+K_3K_7C_{22}\gamma_5\xi_2+K_1K_6K_7\xi_2^2+biK_6K_8s\xi_2^2+2b^2K_6K_7\kappa\xi_2^2\\
&+2b^2K_1K_7C_{22}\xi_2^2+2b^3iK_8sC_{22}\xi_2^2+b^4K_7\kappa C_{22}\xi_2^2-2b^2K_7s\omega T_0\beta_1\xi_2^2-2b^4i^2s\omega B_3T_0\beta_1\xi_2^2-K_3K_7s\xi\chi_1-b^2i^2K_3s\xi B_3\chi_1\\
&+b^2i^2K_3s\xi C_{22}\chi_1-biK_8\xi C_{22}\gamma_4\chi_1+K_7\xi\omega T_0\beta_1\gamma_4\chi_1+b^2i^2\xi\omega B_3T_0\beta_1\gamma_4\chi_1+K_2K_7s\xi_2\chi_1+b^2i^2K_2sB_3\xi_2\chi_1\\
&-b^2i^2K_2sC_{22}\xi_2\chi_1+biK_8C_{22}\gamma_5\xi_2\chi_1-K_7\omega T_0\beta_1\gamma_5\xi_2\chi_1-b^2i^2\omega B_3T_0\beta_1\gamma_5\xi_2\chi_1+K_4K_7\kappa\chi_1^2+K_1K_7\xi\chi_1^2-b^2i^2K_6\kappa\xi\chi_1^2\\
&+b^2i^2K_4\kappa B_3\chi_1^2+b^2i^2K_1\xi B_3\chi_1^2-b^2i^2K_4\kappa C_{22}\chi_1^2-b^2i^2K_1\xi C_{22}\chi_1^2-K_2K_7s\xi_1\chi_2-b^2i^2K_2sB_3\xi_1\chi_2\\
&+b^2i^2K_2sC_{22}\xi_1\chi_2-biK_8C_{22}\gamma_5\xi_1\chi_2+K_7\omega T_0\beta_1\gamma_5\xi_1\chi_2+b^2i^2\omega B_3T_0\beta_1\gamma_5\xi_1\chi_2+K_3K_7s\xi_2\chi_2+b^2i^2K_3sB_3\xi_2\chi_2-b^2i^2K_3sC_{22}\xi_2\chi_2\\
&+biK_8C_{22}\gamma_4\xi_2\chi_2-K_7\omega T_0\beta_1\gamma_4\xi_2\chi_2-b^2i^2\omega B_3T_0\beta_1\gamma_4\xi_2\chi_2-2K_7\kappa\varpi\chi_1\chi_2-2b^2i^2\kappa\varpi B_3\chi_1\chi_2+2b^2i^2\kappa\varpi C_{22}\chi_1\chi_2\\
&-2K_1K_7\xi_2\chi_1\chi_2+2b^2i^2K_6\kappa\xi_2\chi_1\chi_2-2b^2K_7\kappa\xi_2\chi_1\chi_2-2b^2i^2K_1B_3\xi_2\chi_1\chi_2-2b^4i^2\kappa B_3\xi_2\chi_1\chi_2+2b^2i^2K_1C_{22}\xi_2\chi_1\chi_2\\
&+2b^4i^2\kappa C_{22}\xi_2\chi_1\chi_2+K_5K_7\kappa\chi_2^2+b^2i^2K_5\kappa B_3\chi_2^2-b^2i^2K_5\kappa C_{22}\chi_2^2+K_1K_7\xi_1\chi_2^2-b^2i^2K_6\kappa\xi_1\chi_2^2+b^2i^2K_1B_3\xi_1\chi_2^2\\
&-b^2i^2K_1C_{22}\xi_1\chi_2^2+biB_2(K_5K_8s\xi-2K_8s\varpi\xi_2-2b^2K_8s\xi_2^2+biB_3(-K_4K_5\kappa-K_1K_5\xi+\kappa\varpi^2-\left(K_1K_4+K_2\gamma_5\right)\xi_1\\
&+2K_1\varpi\xi_2+2b^2\kappa\varpi\xi_2+K_3\gamma_5\xi_2+2b^2K_1\xi_2^2+b^4\kappa\xi_2^2+\gamma_4\left(-K_3\xi+K_2\xi_2\right))-biK_3s\xi\chi_1+K_8\xi\gamma_4\chi_1+biK_2s\xi_2\chi_1-K_8\gamma_5\xi_2\chi_1\\
&+biK_4\kappa\chi_1^2+biK_1\xi\chi_1^2+biK_3s\xi_2\chi_2-K_8\gamma_4\xi_2\chi_2-2bi\kappa\varpi\chi_1\chi_2-2biK_1\xi_2\chi_1\chi_2-2b^3i\kappa\xi_2\chi_1\chi_2+K_2K_6\gamma_4\xi_2\\
&+K_3K_6\gamma_5\xi_2+2b^2K_1K_6\xi_2^2+b^4K_6\kappa\xi_2^2+C_{22}(\gamma_4\left(-K_3K_4+K2\varpi+b^2K_2\xi_2\right)+\gamma_5\left(-K_2K_5+K_3\varpi+b^2K_3\xi_2\right)\\
&+K_1(-K_4K_5+\varpi^2+2b^2\varpi\xi_2+b^4\xi_2^2))-K_3K_4s\chi_1+K_2s\varpi\chi_1+b^2K_2s\xi_2\chi_1+K_1K_4\chi_1^2+K_2\gamma_5\chi_1^2\\
&-K_2K_5s\chi_2+K_3s\varpi\chi_2+b^2K_3s\xi_2\chi_2-2K_1\varpi\chi_1\chi_2-K_2\gamma_4\chi_1\chi_2-K_3\gamma_5\chi_1\chi_2-2b^2K_1\xi_2\chi_1\chi_2+K_1K_5\chi_2^2+K_3\gamma_4\chi_2^2+\omega T_0\beta_1\\
&(K_4K_5s-s\varpi^2-b^4s\xi_2^2-\varpi\gamma_5\chi_1+K_5\gamma_5\chi_2+\gamma_4\left(K_4\chi_1-\varpi\chi_2\right)-b^2\xi_2\left(2s\varpi+\gamma_5\chi_1+\gamma_4\chi_2\right)))),
\end{aligned}$$

$$
\begin{aligned}
C_5 = (& K_4K_5K_6K_7\kappa + K_1K_5K_6K_7\xi + biK_5K_6K_8s\xi - K_6K_7\kappa\varpi^2 + K_1K_4K_5K_7C_{22} + biK_4K_5K_8sC_{22} - K_1K_7\varpi^2C_{22} \\
& - biK_8s\varpi^2C_{22} - K_4K_5K_7s\omega T_0\beta_1 + K_7s\varpi^2\omega T_0\beta_1 - b^2i^2K_4K_5s\omega B_3T_0\beta_1 + b^2i^2s\varpi^2\omega B_3T_0\beta_1 + K_3K_6K_7\xi\gamma_4 + K_3K_4K_7C_{22}\gamma_4 \\
& - K_2K_7\varpi C_{22}\gamma_4 + K_2K_5K_7C_{22}\gamma_5 - K_3K_7\varpi C_{22}\gamma_5 + K_1K_4K_6K_7\xi_1 + biK_4K_6K_8s\xi_1 + K_2K_6K_7\gamma_5\xi_1 - 2K_1K_6K_7\varpi\xi_2 \\
& - 2biK_6K_8s\varpi\xi_2 - 2b^2K_6K_7\kappa\varpi\xi_2 - 2b^2K_1K_7\varpi C_{22}\xi_2 - 2b^3iK_8s\varpi C_{22}\xi_2 + 2b^2K_7s\varpi\omega T_0\beta_1\xi_2 + 2b^4i^2s\varpi\omega B_3T_0\beta_1\xi_2 \\
& - K_2K_6K_7\gamma_4\xi_2 - b^2K_2K_7C_{22}\gamma_4\xi_2 - K_3K_6K_7\gamma_5\xi_2 - b^2K_3K_7C_{22}\gamma_5\xi_2 - 2b^2K_1K_6K_7\xi_2^2 - 2b^3iK_6K_8s\xi_2^2 - b^4K_6K_7\kappa\xi_2^2 \\
& - b^4K_5K_1K_7C_{22}\xi_2^2 - b^5iK_8sC_{22}\xi_2^2 + b^4K_7s\omega T_0\beta_1\xi_2^2 + b^6i^2s\omega B_3T_0\beta_1\xi_2^2 + K_6B_1(\gamma_4\left(K_3K_4 - K_2\varpi - b^2K_2\xi_2\right) \\
& + \gamma_5\left(K_2K_5 - K_3\varpi - b^2K_3\xi_2\right) + K_1\left(K_4K_5 - \varpi^2 - 2b^2\varpi\xi_2 - b^4\xi_2^2\right)) + K_3K_4K_7s\chi_1 - b^2i^2K_3K_6s\xi\chi_1 - K_2K_7s\varpi\chi_1 \\
& + b^2i^2K_3K_4sB_3\chi_1 - b^2i^2K_2s\varpi B_3\chi_1 - b^2i^2K_3K_4sC_{22}\chi_1 + b^2i^2K_2s\varpi C_{22}\chi_1 + biK_6K_8\xi\gamma_4\chi_1 + biK_4K_8C_{22}\gamma_4\chi_1 - K_4K_7\omega T_0\beta_1\gamma_4\chi_1 \\
& - b^2i^2K_4\omega B_3T_0\beta_1\gamma_4\chi_1 - biK_8\varpi C_{22}\gamma_5\chi_1 + K_7\varpi\omega T_0\beta_1\gamma_5\chi_1 + b^2i^2\varpi\omega B_3T_0\beta_1\gamma_5\chi_1 + b^2i^2K_2K_6s\xi_2\chi_1 - b^2K_2K_7s\xi_2\chi_1 \\
& - b^4i^2K_2sB_3\xi_2\chi_1 + b^4i^2K_2sC_{22}\xi_2\chi_1 - biK_6K_8\gamma_5\xi_2\chi_1 - b^3iK_8C_{22}\gamma_5\xi_2\chi_1 + b^2K_7\omega T_0\beta_1\gamma_5\xi_2\chi_1 + b^4i^2\omega B_3T_0\beta_1\gamma_5\xi_2\chi_1 - \\
& K_1K_4K_7\chi_1^2 + b^2i^2K_4K_6\kappa\chi_1^2 + b^2i^2K_1K_6\xi\chi_1^2 - b^2i^2K_1K_4B_3\chi_1^2 + b^2i^2K_1K_4C_{22}\chi_1^2 - K_2K_7\gamma_5\chi_1^2 - b^2i^2K_2B_3\gamma_5\chi_1^2 \\
& + b^2i^2K2C_{22}\gamma_5\chi_1^2 + K_2K_5K_7s\chi_2 - K_3K_7s\varpi\chi_2 + b^2i^2K_2K_5sB_3\chi_2 - b^2i^2K_3s\varpi B_3\chi_2 - b^2i^2K_2K_5sC_{22}\chi_2 + b^2i^2K_3s\varpi C_{22}\chi_2 \\
& - biK_8\varpi C_{22}\gamma_4\chi_2 + K_7\varpi\omega T_0\beta_1\gamma_4\chi_2 + b^2i^2\varpi\omega B_3T_0\beta_1\gamma_4\chi_2 + biK_5K_8C_{22}\gamma_5\chi_2 - K_5K_7\omega T_0\beta_1\gamma_5\chi_2 - b^2i^2K_5\omega B_3T_0\beta_1\gamma_5\chi_2 \\
& - b^2i^2K_2K_6s\xi_1\chi_2 + biK_6K_8\gamma_5\xi_1\chi_2 + b^2i^2K_3K_6s\xi_2\chi_2 - b^2K_3K_7s\xi_2\chi_2 - b^4i^2K_3sB_3\xi_2\chi_2 + b^4i^2K_3sC_{22}\xi_2\chi_2 - biK_6K_8\gamma_4 \\
& \xi_2\chi_2 - b^3iK_8C_{22}\gamma_4\xi_2\chi_2 + b^2K_7\omega T_0\beta_1\gamma_4\xi_2\chi_2 + b^4i^2\omega B_3T_0\beta_1\gamma_4\xi_2\chi_2 + 2K_1K_7\varpi\chi_1\chi_2 - 2b^2i^2K_6\kappa\varpi\chi_1\chi_2 + 2b^2i^2K_1\varpi B_3\chi_1\chi_2 \\
& - 2b^2i^2K_1\varpi C_{22}\chi_1\chi_2 + K_2K_7\gamma_4\chi_1\chi_2 + b^2i^2K_2B_3\gamma_4\chi_1\chi_2 - b^2i^2K_2C_{22}\gamma_4\chi_1\chi_2 + K_3K_7\gamma_5\chi_1\chi_2 + b^2i^2K_3B_3\gamma_5\chi_1\chi_2 \\
& - b^2i^2K_3C_{22}\gamma_5\chi_1\chi_2 - 2b^2i^2K_1K_6\xi_2\chi_1\chi_2 + 2b^2K_1K_7\xi_2\chi_1\chi_2 - 2b^4i^2K_6\kappa\xi_2\chi_1\chi_2 + 2b^4i^2K_1B_3\xi_2\chi_1\chi_2 - 2b^4i^2K_1C_{22}\xi_2\chi_1\chi_2 \\
& - K_1K_5K_7\chi_2^2 + b^2i^2K_5K_6\kappa\chi_2^2 - b^2i^2K_1K_5B_3\chi_2^2 + b^2i^2K_1K_5C_{22}\chi_2^2 - K_3K_7\gamma_4\chi_2^2 - b^2i^2K_3B_3\gamma_4\chi_2^2 + b^2i^2K_3C_{22}\gamma_4\chi_2^2 \\
& + b^2i^2K_1K_6\xi_1\chi_2^2 - biB_2(K_4K_5K_8s - K_8s\varpi^2 - b^4K_8s\xi_2^2 + biB_3(\gamma_4(-K_3K_4 + K_2\varpi + b^2K_2\xi_2) + \gamma_5\left(-K_2K_5 + K_3\varpi + b^2K_3\xi_2\right) \\
& + K_1\left(-K_4K_5 + \varpi^2 + 2b^2\varpi\xi_2 + b^4\xi_2^2\right)) - biK_3K_4s\chi_1 + biK_2s\varpi\chi_1 + K_4K_8\gamma_4\chi_1 - K_8\varpi\gamma_5\chi_1 + biK_1K_4\chi_1^2 + biK_2\gamma_5\chi_1^2 \\
& - biK_2K_5s\chi_2 + biK_2s\varpi\chi_2 - K_8\varpi\gamma_4\chi_2 + K_5K_8\gamma_5\chi_2 - 2biK_1\varpi\chi_1\chi_2 - biK_2\gamma_4\chi_1\chi_2 - biK_3\gamma_5\chi_1\chi_2 + biK_1K_5\chi_2^2 \\
& + biK_3\gamma_4\chi_2^2 + b^2\xi_2(-2K_8s\varpi + \left(biK_3s - K_8\gamma_4\right)\chi_2 + \chi_1\left(-K_8\gamma_5 + bi\left(K_2s - 2K_1\chi_2\right)\right)))),
\end{aligned}
$$

$$
\begin{aligned}
C_6 = K_6(& -K_1K_4K_5K_7 - biK_4K_5K_8s + K_1K_7\varpi^2 + biK_8s\varpi^2 + 2b^2K_1K_7\varpi\xi_2 + 2b^3iK_8s\varpi\xi_2 + b^4K_1K_7\xi_2^2 + b^5iK_8s\xi_2^2 \\
& + b^2i^2K_3K_4s\chi_1 - b^2i^2K_2s\varpi\chi_1 - b^4i^2K_2s\xi_2\chi_1 - b^2i^2K_1K_4\chi_1^2 + b^2i^2K_2K_5s\chi_2 - b^2i^2K_3s\varpi\chi_2 - b^4i^2K_3s\xi_2\chi_2 \\
& + 2b^2i^2K_1\varpi\chi_1\chi_2 + 2b^4i^2K_1\xi_2\chi_1\chi_2 - b^2i^2K_1K_5\chi_2^2 + \gamma_5(-K_2K_5K_7 + K_3K_7\varpi - b^2i^2K_2\chi_1^2 + b^2\xi_2\left(K_3K_7 + biK_8\chi_1\right) \\
& - biK_5K_8\chi_2 + bi\chi_1\left(K_8\varpi + biK_3\chi_2\right)) + \gamma_4(-K_3K_4K_5K_7 + K_2K_7\varpi + biK_8\varpi\chi_2 - b^2i^2K_3\chi_2^2 \\
& + bi\chi_1\left(-K_4K_8 + biK_2\chi_2\right) + b^2\xi_2\left(K_2K_7 + biK_8\chi_2\right))),
\end{aligned}
$$

Where: $K_1 = \omega\rho c_v\left(1+\tau_0\omega\right) - \kappa b^2$, $K_2 = \omega T_0\gamma_5\left(1+\tau_0\omega\right)$, $K_3 = \omega T_0\gamma_4\left(1+\tau_0\omega\right)$, $K_4 = \gamma_2\omega^2 - \gamma_3 - b^2\xi$,
$K_5 = \omega_o - \gamma_1\omega^2 - \xi_1 b^2$, $K_6 = B_1b^2 - \rho\omega^2$, $K_7 = C_{11}b^2 - \rho\omega^2$, $K_8 = T_0\beta_1 bi\omega$.